S．重力多体系の高精度計算コードの開発

石岡俊也　　　石橋延夫

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概要

We propose a new method to solve trajectories of point mass by using high order B-spline approximation functions. This method is equivalent to an arbitrary order implicit differential method. We have also estimated an upper limit to numerical and truncation errors. This method is useful in the case of stiff equations.

1 Introduction

We propose a high accuracy method to solve the trajectories of particles with B-spline functions. This higher order B-spline approximation function is made with linear summation of arbitrary order of piecewise polynomial basis functions. By using *de Boor-Cox*'s recurrence formula, we can generate these basis functions with high accuracy and efficiency[1]. The equation of motion is,

$$\ddot{\mathbf{r}}_i = -G\sum_{i\neq j} m_j \frac{\mathbf{r}_{ij}}{r_{ij}^3}. \qquad (i \neq j). \tag{1}$$

where, $\mathbf{r}_i$ is a trajectory of the i-th particles, m_i is the mass of the particle, and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, r_{ij} being the distance between the particles. Also $\ddot{\mathbf{r}}_i$ means second time derivative of the i-th trajectory of the particle.

We define the approximation function for each i-th trajectory $\mathbf{r}_i(t)$ of particle by n spline basis functions $B_{j\{il\},k}(l = x, y, z)$ with $(k-1)$-th order, and n expansion coefficients $C_{j\{il\}}$, $\hat{r}_{\{il\}}(t) \equiv \sum_{j=1}^{n} B_{j\{il\},k} C_{j\{il\}}$ $i = 1, \cdots, N$ $l = x, y, z$. (We write $B_{n,k}$ as B_n here.) The expansion coefficient $C_{j\{il\}}$ is obtained uniquely by giving n conditions for the trajectory of i-th particles and each x, y, z. From equation of motion (1), i-th approximation function $\hat{r}_i(t_n)$ at t_n should satisfy the following equation

$$\mathbf{F}_{\{il\}}(t_n) = \ddot{\hat{\mathbf{r}}}_i(t_n) + G\sum_{i\neq j} m_j \frac{\hat{\mathbf{r}}(t_n)_{ij}}{\hat{r}_{ij}(t_n)^3} = \mathbf{0} \qquad i \neq j. \tag{2}$$

at all t_n. The derivative of $\mathbf{F}$ at t_n by $C_{j\{il\}}$ in equation (2) is,

$$J_{n\{il\}} = \frac{\partial F_{\{il\}}}{\partial C_{n\{il\}}} = \frac{\mathrm{d}F_{\{il\}}}{\mathrm{d}\hat{r}_{\{il\}}}\underbrace{\frac{\partial \hat{r}}{\partial C_{n\{il\}}}}_{=B_{n\{il\}}(t_n)} + \frac{\mathrm{d}F_{\{il\}}}{\mathrm{d}\ddot{\hat{r}}_{\{il\}}}\underbrace{\frac{\partial \ddot{\hat{r}}_{\{il\}}}{\partial C_{n\{il\}}}}_{=\ddot{B}_{n\{il\}}(t_n)} \qquad i = 3, \cdots, N, l = x, y, z. \tag{3}$$

Thus we can get numerical solutions by using multi-dimensional Newton method or Gauss-Seidel like iteration methods.

2 Error detection and calculations

The truncation errors is estimated by

$$||\hat{r} - r|| \le \left(\frac{\delta t}{2}\right)^{\max(k+1,l)} \{1 + ||\mathcal{L}^k||\}\, ||r^{(k+1)}||. \tag{4}$$

where $\hat{r}$ is an approximation function and δt is a distance between neighboring knot points, l is a continuity of function $f \in C^l$, and $||\mathcal{L}^m||$ denotes the Lebesgue constant given in Ref. [6],[7]. $\mathcal{L} = 1 \sim 10$ in this cases. We obtain

$$\frac{||\Delta \mathbf{r}||}{||\mathbf{r}||} \le \frac{p_B p_J}{1-\varepsilon_J}\left(\frac{||\Delta J||}{||J||} + \frac{||\delta \mathbf{F}||}{||\mathbf{F}||}\right) + p_B\frac{||\delta \mathbf{C}||}{||\mathbf{C}||} + \varepsilon_B\left\{\frac{p_J}{1-\varepsilon_J}\left(\frac{||\Delta J||}{||J||} + \frac{||\delta \mathbf{F}||}{||\mathbf{F}||}\right) + \frac{||\delta \mathbf{C}||}{||\mathbf{C}||}\right\}, \tag{5}$$

where Δ denotes numerical error and δ denotes round off error of vectors and matrices, p_B and p_J are the condition numbers of matrix B and J, $\varepsilon_B = p_B\frac{||\Delta B||}{||B||}$, $\varepsilon_J = p_J\frac{||\Delta J||}{||J||}$. $p_B p_J$ is a dominant factor of numerical errors. We only have to use δt properly according to the size of $\frac{\mathrm{d}^2 r(u)}{\mathrm{d}t^2}$. Division points $T_{L1}, T_{L2}, \cdots$, are determined so that the interval (or one unit) contains 64 basis functions. One can cope with the stiff equations by using finer mesh point calculation where the second derivative is large as shown in next table.

L1	T_1							
L2	T_1				T_2			
L3	T_1		T_2		T_3		T_4	
L4	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
⋮								

In the case of the sun, the earth (one year), the moon (a month), and Pluto (248 years), it is necessary to calculate all in a usual method. When this method is required, the division points are enough as shown in the following table.

	cycle	Level	calculation points
moon	1month	L10	$(1024) \times 64$
earth	1year	L6	$(64) \times 64$
Pluto	248years	L1	64

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