## G. DUAL–BIMODULES AND THE PROPERTY AB5*

By

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Let $R$ and $S$ be rings with identity and ${}_RQ_S$ an $(R,S)$-bimodule. Then there exists an anti-lattice homomorphism $\theta$ from the submodule lattice of ${}_RR$ into that of $Q_S$ given by

$$\theta(A) = r_Q(A) \text{ for any left ideal } A \text{ of } R,$$

where $r_Q(A)$ denotes the right annihilator of $A$ in $Q$. Similarly we shall denote by $\ell_R(Q')$ the left annihilator in $R$ of a submodule $Q'$ of $Q_S$.

In this case, $\theta$ injective means that

$$\ell_R r_Q(A) = A \text{ for every left ideal } A \text{ of } R$$

and $\theta$ surjective means that

$$r_Q \ell_R(Q') = Q' \text{ for every submodule } Q' \text{ of } Q_S.$$

We shall call ${}_RQ_S$ a left dual-bimodule [6] if $\theta$ is bijective. A ring $R$ for which ${}_RR_R$ is a left dual-bimodule is called a dual ring [2].

Lemonnier [8] has shown that, for a quasi-injective module $Q_S$ with $R = \mathrm{End}(Q_S)$, $\theta$ is bijective iff it is surjective and $Q_S$ is finitely cogenerated and AB5*. Corresponding to this, it is shown in Section 1 that if ${}_RQ$ is finitely cogenerated, quasi-injective with $S = \mathrm{End}({}_RQ)$, then $\theta$ is bijective iff it is injective and ${}_RR$ is AB5* (Theorem 1.4). Using this it is also shown that ${}_RR$ is AB5* iff the minimal cogenerator ${}_RQ$ for $R$-mod is a left dual-bimodule with $S = \mathrm{End}({}_RQ)$ (Theorem 1.6).

From Lemonnier [8] it follows that if $\theta$ is bijective then both ${}_RR$ and $Q_S$ are AB5*. The converse need not be true in general. However, we can show in Section 2 that $\theta$ is bijective iff ${}_RR$ and $Q_S$ are both AB5*, in case ${}_RQ_S$ defines a quasi-duality in the sense of Kraemer [5] (Theorem 2.3).

In Section 3, using the notion of the relative simple injectivity, we shall give another characterization of the injectivity and of the surjectivity of $\theta$ and show that a ring $R$ is a dual ring iff ${}_RR$ and $R_R$ are both simple injctive and Kasch (Corollary 3.4).

Finally, we shall provide in Section 4 an example to illustrate the results given in this paper.

The detailed version of this paper will be submitted for publication elsewhere.

**1.** We say that an $R$–module $_RM$ satisfies the property AB5* or simply that $_RM$ is AB5* if

$$M' + \cap_{i\in I} M_i = \cap_{i\in I}(M' + M_i)$$

for any submodule $M'$ and any inverse family $\{M_i\}_{i\in I}$ of submodules of $_RM$, where $\{M_i\}_{i\in I}$ an inverse family means that for any pair $i, j \in I$ there exists a $k \in I$ such that $M_k \leq M_i \cap M_j$.

Corresponding to Lemonnier [8, Proposition 7], we have the folloing theorem, which can be seen as a generalization of [8, Corollary 8 and Theorem 13].

**Theorem 1.4.** Let $_RQ$ be finitely cogenerated, quasi-injective with $S = \mathrm{End}(_RQ)$. Then $\theta$ is bijective iff it is injective and $_RR$ is AB5*.

As an application of Theorem 1.4, we have:

**Theorem 1.6.** For a ring $R$ the following conditions are equivalent:

(1) $_RR$ is AB5*.

(2) There exists a cogenerator $_RQ$ for $R$–mod which is a left dual–bimodule with $S = \mathrm{End}(_RQ)$.

(3) The minimal cogenerator $_RQ$ for $R$–mod is a left dual-bimodule with $S = \mathrm{End}(_RQ)$.

**2.** Let $_RQ_S$ be an $(R, S)$–bimodule. Following Kraemer [5] we say that $Q$ defines a quasi–duality if $Q$ is a faithfully balanced bimodule and if $_RQ$ and $Q_S$ are finitely cogenerated, quasi–injective. Combining Propositions 1.3 and 2.2, we have:

**Theorem 2.3.** If an $(R, S)$–bimodule $_RQ_S$ defines a quasi-duality, then $\theta$ is bijective iff $_RR$ and $Q_S$ are both AB5*.

**3.** In this section, we shall give another characterization of the injectivity and of the surjectivity of $\theta$. To do this, for an $R$–module $_RM$, following Harada [3] we shall call $_RQ$ simple $M$–injective if every $R$–homomorphism from a submodule of $M$ to $Q$ with simple image can be extended to an $R$–homomorphism $M \to Q$. We shall simply call $_RQ$ simple injective if it is simple $R$–injective.

**Proposition 3.1.** Let $_RQ_S$ be an $(R, S)$–bimodule. If $_RQ$ is simple–injective, then the following conditions are equivalent:

(1) $\theta$ is injective.

(2) Every simple left $R$–module can be embedded in $_RQ$.

(3) Every simple left $R$–module isomorphic to a factor module of a left ideal of $R$ can be embedded in ${}_RQ$.

**Proposition 3.2.** Let ${}_RQ_S$ be an $(R, S)$–bimodule with $R = \mathrm{End}(Q_S)$. If $Q_S$ is simple $Q$–injective, then the following conditions are equivalent:

(1) $\theta$ is surjective.

(2) Every factor module of $Q_S$ is $Q$–torsionless.

(3) Every simple right $S$–module isomorphic to a factor module of a submodule of $Q_S$ can be embedded in $Q_S$.

By Propositions 3.1 and 3.2 and [6, Lemma 1.13] we have:

**Theorem 3.3.** Let ${}_RQ_S$ be an $(R, S)$–bimodule with $R = \mathrm{End}(Q_S)$. Assume that $Q_S$ is simple $Q$–injective. Then $\theta$ is bijective iff

(1) ${}_RQ$ is simple injective,

(2) every simple left $R$–module can be embedded to ${}_RQ$ and

(3) every simple right $S$–module isomorphic to a factor module of a submodule of $Q_S$ can be embedded in $Q_S$.

**Corollary 3.4.** A ring $R$ is a dual ring iff ${}_RR$ and $R_R$ are both simple injective and Kasch.

**4.** Finally we shall provide an example ([6, Examples 4.1 and 4.2]) to illustrate the results given in this paper.

Let $p$ be a prime number and let $R = \mathbf{Z}_{(p)} = \{b/a \in \mathbf{Q} \mid (a, b) = 1 \text{ and } (a, p) = 1\}$, where $\mathbf{Q}$ denotes the field of rational numbers. Then $R$ is a commutative local ring with the unique maximal ideal $Rp$ and nonzero proper ideals of $R$ are exhausted by $Rp^n, n > 0$. The quotient module $\mathbf{Q}/R$ is an $(R, R)$–bimodule and the only nonzero proper submodules are those of the form $Rp^{-n}/R$ for some $n > 0$. Since $\mathbf{Q}/R$ is a left deal-bimodule,

(1) ${}_RR$ is AB5*.

Now let $Q = Rp^{-n}/R$ and $\bar{R} = R/Rp^n$. Then $Q$ is an $(R, \bar{R})$–bimodule and nonzero proper submodules of $Q_{\bar{R}}$ are exhausted by $Rp^{-i}/R, 1 \leq i \leq n-1$. Hence, there is no lattice isomorphism between the submodule lattices of ${}_RR$ and $Q_{\bar{R}}$. Therefore ${}_RQ_{\bar{R}}$ is not a left dual-bimodule. However,

(2) $Q_R$ is AB5*.

(3) Both ${}_RQ$ and $Q_R$ are finitely cogenerated quasi-injective and $\bar{R} = \mathrm{End}\ ({}_RQ)$ but not $R = \mathrm{End}(Q_{\bar{R}})$.

(4) $\theta$ is surjective, but not injective.

(5) ${}_RQ$ is Kasch, but not simple injective.

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