

3. On the Inseparable Degree of the Gauss Map of Higher Order for Space Curves

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(Communicated by Heisuke HIRONAKA, M. J. A., Jan. 13, 1992)

Abstract: Let X be a curve non-degenerate in a projective space P^N defined over an algebraically closed field of positive characteristic p , consider the Gauss map of order m defined by the osculating m -planes at general points of X , and denote by $\{b_j\}_{0 \leq j \leq N}$ the orders of X . We prove that the inseparable degree of the Gauss map of order m is equal to the highest power of p dividing b_{m+1} .

Key words: Space curve, Gauss map, inseparable degree.

0. Introduction. Let X be an irreducible curve in a projective space P^N defined over an algebraically closed field k of positive characteristic p , C the normalization of X , and $\iota: C \rightarrow P^N$ the natural morphism. Denote by $\iota^{(m)}: C \rightarrow G(P^N, m)$ the Gauss map of order m defined by the osculating m -planes of X , where $G(P^N, m)$ is a Grassmann manifold of m -planes in P^N . Assume that X is non-degenerate in P^N , and let $\{b_j\}_{0 \leq j \leq N}$ be the orders of ι . The purpose of this short note is to prove

Theorem. *The inseparable degree of $\iota^{(m)}$ is the highest power of p dividing b_{m+1} .*

In case of $m=1$, Theorem is known: For $N=2$, see [4, Proposition 4.4]; for a general N , see [5, Remark below Corollary 2.3], [3, Proposition 4]. A corollary to this result will give a generalization of [5, Theorem 2.1] (see Corollary below).

In case of $m=N-1$, Theorem coincides with a result of A. Hefez and N. Kakuta, announced in [1]. Although it has not been published yet, according to Hefez [2], their proof for the theorem is similar in spirit to ours (precisely speaking, of the first version), but not identical. Hefez and Kakuta moreover found

Theorem (Hefez-Kakuta). *Denote by $C^{(m)}X$ the conormal variety of order m , and by $X^{*(m)}$ the m -dual. Then the inseparable degree of the natural morphism $C^{(m)}X \rightarrow X^{*(m)}$ is equal to the highest power of p dividing b_{m+1} .*

This result is stated as a theorem in [2] without proof.

We finally mention that this Theorem of Hefez and Kakuta is deduced also from our theorem and a result in [6] that $C^{(m)}X \rightarrow X^{*(m)}$ has the same inseparable degree as $\iota^{(m)}$, which is proved directly without going through

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the orders.

1. Notations and terminology. Denote by V the image of the natural map $\iota^*: H^0(P^N, \mathcal{O}_{P^N}(1)) \rightarrow H^0(C, \iota^*\mathcal{O}_{P^N}(1))$. For a general point P in C , choose a trivialization $\iota^*\mathcal{O}_{P^N}(1)_P \simeq \mathcal{O}_{C,P}$, and consider V as a subset of the rational function field $K(C)$. Then, taking a suitable basis $\{x_0, \dots, x_N\}$ for V , one can expand x_i as follows:

$$(*) \quad x_i = \sum_{n \geq b_i} a_{i,n} t^n$$

in $\hat{\mathcal{O}}_{C,P} \simeq k[[t]]$ with $a_{i,b_i} \neq 0$, for some non-negative integers $b_0 < b_1 < \dots < b_N$, where t is a local parameter of C at P . The orders of ι are defined to be $\{b_j\}_{0 \leq j \leq N}$ and the osculating m -plane of ι at P , denoted by $T_P^{(m)}$, is a linear space determined by $X_{m+1} = \dots = X_N = 0$, where $(X_0 : \dots : X_N)$ is a homogeneous coordinate of P^N corresponding to the basis $\{x_i\}_{0 \leq i \leq N}$. The Gauss map of order m of ι , denoted by $\iota^{(m)}$, is defined as the elimination of the base locus of the rational map $C \rightarrow G(P^N, m)$, sending P to $T_P^{(m)}$ for general points P of C , which is locally given by a matrix, $(D_t^{(b_j)} x_i)_{0 \leq i \leq N, 0 \leq j \leq m}$, where $D_t^{(k)}$ is the Hasse differential operator of order k with respect to the parameter t .

For full details of the above, we refer to [7, §§4, 5], [9, §1].

2. Proof of Theorem. Let q be the highest power of p dividing b_{m+1} , and denote by c the quotient:

$$b_{m+1} = cq.$$

Since $(c-1)q$ is non-negative and p -adically less than b_{m+1} , according to [8, Satz 6], it is an order of ι , say b_k , with $0 \leq k \leq m$.

Lemma. (a) $D_t^{(b_j)} x_i$ is a unit if $j=i$, and is not if $j < i$.

(b) $D_t^{(b_j)} x_i$ is of order at least q if $j \leq m < i$.

(c) $D_t^{(b_k)} x_{m+1}$ is of order exactly q .

Proof. We first note that

$$(*) \quad D_t^{(b_j)} x_i = \sum_{n \geq b_i} a_{i,n} \binom{n}{b_j} t^{n-b_j}$$

for arbitrary i, j . Thus, (a) follows from this (*).

For (b), suppose the contrary: According to (*), there would exist an integer $n \geq b_i$ such that

$$(**) \quad \binom{n}{b_j} \not\equiv 0 \pmod{p}, \text{ and}$$

$$(***) \quad n - b_j < q.$$

Dividing by q , writing n and b_j as $n = n'q + n''$, $b_j = b'q + b''$ with $0 \leq n'', b'' < q$, we have

$$\binom{n}{b_j} = \binom{n'q + n''}{b'q + b''} \equiv \binom{n'}{b'} \binom{n''}{b''} \pmod{p}.$$

In particular, $n'' \geq b''$ because of (**). On the other hand, $n' \geq c > b'$ since $n \geq b_i \geq b_{m+1} > b_j$ and $b_{m+1} = cq$. Therefore, it follows

$$n - b_j = (n' - b')q + (n'' - b'') \geq q,$$

which contradicts to (***) .

Finally for (c), the first term of the summation in (*) for $D_t^{(b_k)} x_{m+1}$ is

$$\binom{b_{m+1}}{b_k} t^{b_{m+1}-b_k},$$

and this is not zero: In fact, we have $\binom{b_{m+1}}{b_k} \equiv \binom{c}{c-1} = c \not\equiv 0 \pmod{p}$; because $b_{m+1} = cq$, $b_k = (c-1)q$, and c is coprime to p . This proves (c) since $b_{m+1} - b_k = q$.

Now we prove Theorem. Let Δ_{i_0, \dots, i_m} be the submatrix of $(D_t^{(b_j)} x_i)_{0 \leq i \leq N, 0 \leq j \leq m}$, consisting of the rows i_0, \dots, i_m , where we start with zero to count those rows. Since the rational map sending P to $T_P^{(m)}$ is determined by the minors,

$$(\det \Delta_{i_0, \dots, i_m})_{0 \leq i_0 < \dots < i_m \leq N}$$

via the Plücker embedding of $G(P^N, m)$, it suffices to show:

- (1) $\det \Delta_{0, \dots, m}$ is a unit;
- (2) $\det \Delta_{i_0, \dots, i_m}$ is of order at least q unless $(i_0, \dots, i_m) = (0, \dots, m)$;
- (3) $\det \Delta_{0, 1, \dots, k-1, k+1, \dots, m+1}$ is of order exactly q .

Proof. (1) It follows from Lemma (a) that $\Delta_{0, \dots, m} \pmod{t}$ is lower triangular and each entry on the diagonal is a unit. So $\det \Delta_{0, \dots, m}$ is also a unit as required.

(2) If $(i_0, \dots, i_m) \neq (0, \dots, m)$, then $i_m > m$. It follows from Lemma (b) that each entry in the last row Δ_{i_0, \dots, i_m} has order at least q , which implies (2).

(3) Consider the matrix

$$\Delta_{0, \dots, k-1, m+1, k+1, \dots, m}$$

instead of $\Delta_{0, \dots, k-1, k+1, \dots, m+1}$. Ignoring the k -th row, by the similar way to (1) we see that this is equal to a lower triangular matrix mod t and the entries on the diagonal are all units. For the k -th row, according to (b) and (c) in Lemma, every entry has order at least q and the one on the diagonal has order exactly q . We thus conclude that its determinant is of order exactly q , and this completes the proof of (3).

We have thus proved Theorem.

Corollary (Generic Projection). *The inseparable degree of $\iota^{(m)}$ is invariant under a projection of P^N from a generic point, where $1 \leq m \leq N-2$.*

Proof. Since the order b_{m+1} is invariant under a generic projection, the result thus follows from Theorem.

Acknowledgements. The first author was partially supported by Grant-in-Aid for Scientific Research (C) (No. 02804002), The Ministry of Education, Science and Culture, Japan. The second author was partially supported by Grant-in-Aid for Encouragement of Young Scientists (No. 02740053), The Ministry of Education, Science and Culture, Japan.

Addendum. After the draft was written up, we received a preprint "On the geometry of non-classical curves" by A. Hefez and N. Kakuta, which relates to the work mentioned in Introduction.

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