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PLANE CURVES WITH ALIGNED CONDUCTORS

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Abstract

An invariant toward a classification of curves X with a birationally very ample and special invertible sheaf L is proposed, and an extremal pair (X, L) by means of the invariant is a smooth plane curve X with $\mathcal{O}_X(1)$ corresponding line sections and vice versa. For pairs (X, L) next to extremal ones, all but one exception is characterized by the image $\phi_L(X)$ to be a plane singular curve with aligned conductors. The heart of the paper is to study such a singular plane curve with its normalization involving projective geometry and the theory of adjoint curves.

0. Introduction

One of the essential parts in the proof of Castelnuovo's genus bound on a nondegenerate curve Y in $\mathbb{P}^r$ with $r \geq 3$ is the following observation: Let $X \xrightarrow{\psi} Y$ be the normalization of Y, and $L := \psi^* \mathcal{O}_Y(1)$. Then we have $h^1(X, L^m) = 0$ if $m \geq \left[\dfrac{d-2}{r-1}\right]$, where $d := \deg L$ is the degree of L and $r := r(L)$ is the projective dimension of the linear system $|L|$ corresponding to $H^0(X, L)$ (see [2, III, Section 2] or [11, III, Section

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2, Theorem 1]). So it seems natural to study the quantity

$$m(L) := \min\{m \mid h^1(X, L^m) = 0\}$$

for a birationally very ample and special invertible sheaf L on a smooth curve X. Here a *birationally very ample* invertible sheaf L means that the rational map $\phi_L : X \to \mathbb{P}^r$ corresponding to $|L|$ is birational to the closure of its image, which may be a morphism or it may not.

Although we know the estimation $m(L) \leq \left\lceil \frac{d-2}{r-1} \right\rceil$ if $r \geq 3$, the following fact may be interesting: *We have* $m(L) \leq d - r$ *for any birationally very ample invertible sheaf* L *on a smooth curve* X, *and equality holds if and only if the curve is a smooth plane curve and* $L = \mathcal{O}_X(1)$ *which is the invertible sheaf corresponding to line sections of* $X \subset \mathbb{P}^2$. We give a proof of the assertion in Section 1.

Motivated by the fact, we propose classifying the birationally very ample and special invertible sheaves according to the quantity

$$\theta(L) := \deg L - r(L) - m(L),$$

which has been already introduced implicitly in [7, Section 4], and we describe the birationally very ample and special invertible sheaf L with $\theta(L) = 1$ in Section 2.

In this description, we meet a plane curve Y whose conductors lie on a line (see, Theorem 2.7). If the invariant $\theta(L)$ is promising, there must be a similarity between a plane curve with aligned conductors and a smooth one because they are just neighbours in the sense of the invariant. One of the remarkable properties of a smooth plane curve X of degree $d \geq 4$ is the variety of special divisor $W_d^2(X)$ consists of one point, which corresponds to the linear system cut out by lines. A generalization of the fact to a singular curve of degree d with only a small number of ordinary nodes or cusps as its singularities, which is a neighbour of smooth curves in another sense, was given by Coppens and Kato [4, Theorem 2.4]. As a matter of fact, they didn't say about the net explicitly, but one can easily derive from it the same conclusion on $W_d^2(X)$ for the normalization X of the singular curve under the same assumption in their theorem.

A major part of the paper is devoted to study the linear system on the normalization X which is cut out on Y by lines, especially its uniqueness, for a plane curve Y with aligned conductors.

In order to see the uniqueness, we need a lemma in projective geometry, which is analogous to [2, Appendix A, Exercise 17] and [5, Proposition 1]. We handle the projective geometry in Section 3.

Although the completeness of the linear system is shown at Lemma 2.6, the proof seems somewhat ad hoc, and does not apply to the case of characteristic 2. So we give a more theoretical proof of the fact in Section 4, which is based on a result of the theory of adjoint curves. The result is a generalization of [2, Appendix A, Exercise 24], and we treat the matter in Appendix.

We work over an algebraically closed field k. We do not put any restriction on the characteristic of k except the results of Lemma 1.2, namely Theorem 1.1, Lemma 2.1, Lemma 2.6 and Theorem 2.7, for which we assume that the characteristic of k is not 2.

1. A Characterization of the Smooth Plane Curve

Throughout this paper, X denotes a complete, connected, smooth curve of genus $g \geq 2$ over k. The following is our starting point.

Theorem 1.1. *Assume that the characteristic of k is not 2. Let L be an invertible sheaf of degree d on X with $h^0(X, L) = r + 1$. If L is birationally very ample, then $m(L) \leq d - r$, and equality holds if and only if L is very ample with $r = 2$, that is, $\phi_L(X)$ is a smooth plane curve of degree d.*

To show the theorem, we need Castelnuovo's genus bound in our situation.

Lemma 1.2. *Assume that the characteristic of the ground field k is not 2. Let g be the genus of the normalization of a nondegenerate irreducible curve Y of degree $d \geq 3$ in $\mathbb{P}^3$. Then we have*

$$g \leq \begin{cases} \dfrac{1}{4}d^2 - d + 1 & \text{if } d \text{ is even} \\ \\ \dfrac{1}{4}(d^2 - 1) - d + 1 & \text{if } d \text{ is odd.} \end{cases}$$

Proof. Hartshorne's proof [8, IV, Theorem 6.4] of Castelnuovo's inequality is effective for a nondegenerate space curve Y with the property that a general secant of Y is not a multisecant, even if Y is not smooth.

If Y fails to enjoy the property, then Y must be strange by [8, IV, Proposition 3.8]. (Note that its proof is also effective without the smoothness assumption on Y.) Thanks to Bayer and Hefez [3, Corollary 7.4], we know an upper bound of the genus g of the normalization of a strange curve Y in terms of its degree as follows: Let q be the inseparable degree of the natural morphism from the conormal variety of Y to the dual variety, which is a positive power of the characteristic of k, and let s be the separable degree of the morphism. Then $t := d - sq$ is the remainder of the division of d by q, i.e., $0 \leq t < q$, and we have

$$g \leq \frac{1}{2}(s-1)(d+t-2).$$

Since the characteristic of k is not 2, we have $q \geq 3$, and hence $s \leq \dfrac{d}{3}$. Additionally it is obvious that $t \leq \dfrac{d-1}{2}$. So we have

$$g \leq \frac{1}{2}\left(\frac{d}{3} - 1\right)\left(\frac{3}{2}d - \frac{5}{2}\right),$$

which is more restrictive than the desired inequality.

Proof of Theorem 1.1. If the linear system $|L|$ has a base point Q, then we have $m(L) \leq m(L(-Q))$ and $\deg L(-Q) - r(L(-Q)) < \deg L - r(L)$. So we may assume that $|L|$ has no base points.

Case 1. If $h^1(L) = 0$, then $d - r = g \geq 2$. Hence the strict inequality holds because $m(L) = 1$.

Case 2. Next, we consider the case $h^1(L) > 0$. Assume that $m(L) \geq d - r$. Since L^{m-1} is special, we have

$$2g - 2 \geq (m-1)d \geq (d-r-1)d. \tag{1}$$

Hence, by Clifford's theorem which says $r \leq \frac{d}{2}$, we have $2g - 2 \geq \left(\frac{d}{2} - 1\right)d$, that is,

$$g \geq \frac{1}{4}d^2 - \frac{1}{2}d + 1. \tag{2}$$

We further assume that $r \geq 3$, and consider the image of $\phi_L(X)$ by a generic projection $\mathbb{P}^r \to \mathbb{P}^3$. Applying Castelnuovo's bound to the image curve, we have

$$g \leq \frac{1}{4}d^2 - d + 1. \tag{3}$$

But inequalities (2) and (3) are not compatible. Hence r must be 2, and $g \geq \frac{(d-1)(d-2)}{2}$ by (1). Therefore, the only possibility of the inequality $m(L) \geq d - r$ holding is the case where $\phi_L(X)$ is a smooth plane curve of degree d. In this case, each equality in (1) holds. In particular, $m(L) = d - r$. Except for this case, we have $m(L) < d - r$.

Remark 1.3. The inequality $m(L) \leq d - r$ may not hold without the birationally very ampleness of L. In fact, if M is the invertible sheaf with $|M| = g_2^1$ on a hyperelliptic curve of genus g, then $m(M) = g - 1$.

2. A Quantity Toward the Classification

2.1. Definition of $\theta(L)$

As was mentioned in Introduction, we introduce a quantity for a birationally very ample invertible sheaf L on X:

$$\theta(L) \overset{def}{=} \deg L - r(L) - m(L). \tag{4}$$

The quantity can be also written as

$$\theta(L) = g - h^1(L) - m(L) \tag{5}$$

by the Riemann-Roch theorem. In particular, if $h^1(L) = 0$, then $\theta(L) = g - 1$. As for a special L, we have the following:

Lemma 2.1. *Assume that the characteristic of k is not 2. Let L be a birationally very ample invertible sheaf with $h^1(L) > 0$. Then we have*

$$0 \leq \theta(L) \leq g - 3.$$

Proof. The left-hand inequality is just a part of Theorem 1.1. Since $h^1(L) > 0$ implies $m(L) \geq 2$, we have $\theta(L) \leq g - 3$ by (5).

Example 2.2. For the canonical sheaf ω_X, we have $\theta(\omega_X) = g - 3$; more precisely, $\theta(L) = g - 3$ if and only if $h^1(L) = 1$.

We intend to classify birationally very ample and special invertible sheaves L on a smooth curve by $\theta(L)$, and the characterization of L with $\theta(L) = 0$ has been done in the previous section. So the next task should be to characterize L with $\theta(L) = 1$. To describe the image $\phi_L(X)$ for L with $\theta(L) = 1$, we need some terminology in the theory of singular curves.

2.2. A Plane Curve with Singularities

Let Y be an irreducible curve of degree $d > 1$ in $\mathbb{P}^2$, and $X \xrightarrow{\psi} Y$ the normalization of Y. We always assume that Y has a singular point and denote by $\operatorname{Sing} Y$ the set of singular points of Y. Let

$$\mathcal{C} := \operatorname{Ann}_{\mathcal{O}_Y}(\psi_* \mathcal{O}_X / \mathcal{O}_Y)$$

be the sheaf of conductors of Y. Note that $\mathcal{C}$ is an ideal both of $\mathcal{O}_Y$ and of $\psi_* \mathcal{O}_X$. Obviously, the support of $\mathcal{C}$ on Y is just $\operatorname{Sing} Y$. We denote by D the subscheme $\operatorname{Spec} \mathcal{O}_Y / \mathcal{C}$ of Y, and by C the D-scheme $\operatorname{Spec} \psi_* \mathcal{O}_X / \mathcal{C}$. So we have a Cartesian diagram

$$\begin{array}{ccc} C & \hookrightarrow & X \\ \downarrow & & \downarrow \psi \\ D & \hookrightarrow & Y. \end{array}$$

Let δ be the length of $\psi_*\mathcal{O}_X/\mathcal{O}_Y$, and $\delta_P := \dim_k \psi_*\mathcal{O}_{X,\,P}/\mathcal{O}_{Y,\,P}$. So δ is, by definition, $\sum_{P\in \operatorname{Sing} Y} \delta_P$. An important result of Gorenstein [6, Theorem 10] is

$$\dim_k \psi_*\mathcal{O}_{X,\,P}/\mathcal{C}_P = 2\delta_P; \tag{6}$$

hence the length of D is δ and that of C is 2δ.

Let $\mathcal{K}$ be the constant sheaf of the function field of Y. Then those sheaves are subsheaves of $\mathcal{K}$, and there is a sequence of $\mathcal{O}_Y$-modules:

$$\mathcal{K} \supset \psi_*\mathcal{O}_X \supset \mathcal{O}_Y \supset \mathcal{C}.$$

If we regard D as a generalized divisor on Y, we understand $\mathcal{O}_Y(-D)$ to be $\mathcal{C}$. Since each torsion-free $\mathcal{O}_Y$-module is reflexive [9, Lemma 1.1] and $\mathcal{C} \simeq \mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \mathcal{O}_Y)$, we have $\mathcal{O}_Y(D) = \psi_*\mathcal{O}_X$.

The next lemma is a generalization of [6, Theorem 8].

Lemma 2.3. *For an invertible sheaf M on Y, we have*

$$H^0(Y, M(-D)) \simeq H^0(X, \psi^*M(-C)).$$

Proof. Since $\psi : X \to Y$ is an affine morphism, the sequence

$$0 \to \psi_*(\mathcal{O}_X(-C)) \to \psi_*\mathcal{O}_X \to \psi_*\mathcal{O}_C \to 0$$

is exact. Since $C = \operatorname{Spec} \psi_*\mathcal{O}_X/\mathcal{C}$ is a Y-scheme via ψ, we have $\psi_*\mathcal{O}_C \simeq \psi_*\mathcal{O}_X/\mathcal{C}$. Hence

$$\psi_*(\mathcal{O}_X(-C)) \simeq \mathcal{C} \simeq \mathcal{O}_Y(-D).$$

Thus we have

$$M(-D) \simeq M \otimes \psi_*(\mathcal{O}_X(-C)) \simeq \psi_*(\psi^*M(-C))$$

by the projection formula.

Since Y is in $\mathbb{P}^2$, we have the twisting sheaf $\mathcal{O}_Y(1)$ corresponding to the linear system cut out by lines on Y. Another important result of [6] is that

$$\omega_X \simeq (\psi^* \mathcal{O}_Y(d-3))(-C), \tag{7}$$

where ω_X is the canonical sheaf on X.

In particular, together with (6), we can compute the genus g of X as

$$2g - 2 = d(d-3) - 2\delta;$$

and together with Lemma 2.3, we have

$$H^0(X, \omega_X) \simeq H^0(Y, \mathcal{O}_Y(d-3)(-D)). \tag{8}$$

We add a small historical remark. Gorenstein originally defined δ as

$$h^0(Y, \mathcal{O}_Y(n)) - h^0(Y, \mathcal{O}_Y(n)(-D))$$

for large n [6, p. 431], which means that δ denotes the length of $\mathcal{O}_Y/\mathcal{C}$. Nowadays, however, people use δ for the length of $\psi_* \mathcal{O}_X/\mathcal{O}_Y$ (see, [8, IV, Exercise 1.8] or [10, IV]). Anyhow, no confusion may occur in our case because of (6).

2.3. Characterization of L with $\theta(L) = 1$

Now, we introduce a new class of singular plane curves, which is necessary for the description of L with $\theta(L) = 1$.

Definition 2.4. The sheaf of conductors $\mathcal{C}$ of Y is said to be *aligned* if the scheme of conductors D is a subscheme of a line. In this case, we also say that *the conductors of Y is aligned.*

We can paraphrase the definition in several ways.

Lemma 2.5. *The following conditions are equivalent:*

(i) *the sheaf of conductors $\mathcal{C}$ of Y is aligned;*

(ii) *there exists an injective $\mathcal{O}_Y$-homomorphism $\mathcal{O}_Y(-1) \hookrightarrow \mathcal{C}$;*

(iii) $h^0(Y, \mathcal{O}_Y(1)(-D)) > 0$.

The proof is easy, so we skip it.

Lemma 2.6. *Assume that the characteristic of k is not 2, and the normalization X of a plane curve Y is of genus $g \geq 2$. Let L be the invertible sheaf $\psi^*\mathcal{O}_Y(1)$. If the conductors of Y is aligned, then $r(L) = 2$ and $\theta(L) = 1$.*

Proof. Since $h^0(Y, \mathcal{O}_Y(1)(-D)) > 0$ by Lemma 2.5 and

$$H^0(Y, \mathcal{O}_Y(1)(-D)) \simeq H^0(X, L(-C))$$

by Lemma 2.3, there is an effective divisor E on X such that $L(-C) \simeq \mathcal{O}_X(E)$. On the other hand, since $\omega_X = L^{d-3}(-C)$ by (7), we have $L^{d-4} = \omega_X(-E)$. Hence we have $h^1(L^{d-4}) > 0$, which implies $m(L) = d-3$ because L^{d-3} is the nonspecial invertible sheaf $\omega_X(C)$. Therefore,

$$\theta(L) = d - r(L) - (d-3) = 3 - r(L).$$

Since $\theta(L) \geq 1$ by Theorem 1.1, we have $r(L) \leq 2$, and equality must hold because $r(L) \geq 2$ in general. This completes the proof.

Theorem 2.7. *Assume that the characteristic of k is not 2. Let L be a birationally very ample invertible sheaf on a smooth curve X of genus $g \geq 3$. Then $\theta(L) = 1$ if and only if either*

(1) *X is a nonhyperelliptic curve of genus 4 with $L = \omega_X$, or*

(2) *the linear system $|L|$ is free from base points, $r(L) = 2$, and $\phi_L(X)$ is a singular plane curve with aligned conductors.*

Proof. The "if" part is just Example 2.2 and Lemma 2.6. So we have to prove the "only if" part. Since $g \geq 3$, $\theta(L) = 1$ implies $h^1(L) > 0$. In particular, X is nonhyperelliptic because no special linear system on a hyperelliptic curve is birationally very ample.

Suppose that $|L|$ has a base point. Let B be the base locus of $|L|$. Then we have

$$0 \le \theta(L(-B)) \le \theta(L) - \deg B = 1 - \deg B;$$

therefore, $\deg B = 1$ and $\theta(L(-B)) = 0$. From Theorem 1.1, X is a smooth plane curve with a certain degree, say e, and $L(-B) \simeq \mathcal{O}_X(1)$. Note that $e \ge 4$ because $g \ge 3$. Hence

$$L^{(e-3)} \simeq \mathcal{O}_X(e-3) \otimes \mathcal{O}_X((e-3)B) \simeq \omega_X((e-3)B),$$

which means $L^{(e-3)}$ is nonspecial. Hence we have $m(L) \le e-3$. Note that $\deg L = e+1$ and $r(L) = r(L(-B)) = 2$. So, we have $\theta(L) \ge 2$, which contradicts with our assumption $\theta(L) = 1$. Therefore, $|L|$ has no base points.

Now, we denote by d the degree of L, and by Y the image curve $\phi_L(X)$. Note that $\phi_L : X \to Y$ is the normalization of Y. So, in order to avoid the confusion of notation, we denote by ψ instead of ϕ_L.

Furthermore, we assume that (X, L) is not in case (1). Then we have $r \le \dfrac{d-1}{2}$ by Clifford's theorem because the linear system $|L|$ is neither canonical nor g^1_2. By using the very same argument as we did in Case 2 in the proof of Theorem 1.1, we can show that $r(L) = 2$. In fact, since

$$\begin{aligned} 2g-2 &\ge (m-1)d \\ &= (d-r-2)d \quad (\text{because } d-r-m=1) \\ &\ge \frac{1}{2}d^2 - \frac{3}{2}d, \end{aligned}$$

we have $g \ge \dfrac{1}{4}d^2 - \dfrac{3}{4}d + 1$. On the other hand, if $r \ge 3$, the genus g must be less than or equal to $\dfrac{1}{4}d^2 - d + 1$ by Castelnuovo's bound, which is a contradiction.

In order to show the sheaf of conductors $\mathcal{C}$ to be aligned, we prove that $h^0(Y, \mathcal{O}_Y(1)(-D)) > 0$ (see, Lemma 2.5). Since we already know that $r(L) = 2$ for the L of degree d with $\theta(L) = 1$, we have $m(L) = d - 3$. Hence $h^0(X, \omega_X \otimes L^{-(d-4)}) > 0$. On the other hand, since $\omega_X \otimes L^{-(d-4)} \simeq (\psi^*\mathcal{O}_Y(1))(-\mathcal{C})$ by (7), the positivity means $h^0(X, \psi^*(\mathcal{O}_Y(1))(-\mathcal{C})) > 0$. Hence we can conclude that $h^0(Y, \mathcal{O}_Y(1)(-D)) > 0$ by Lemma 2.3.

Before closing this section, we explain some observations about a singular point of a curve with the conductor at the point lying on a line.

Example 2.8. (a) If $\delta_P = 1$ for every singular point P of Y, the conductors of Y is aligned if and only if those singular points are collinear.

(b) If $\operatorname{Sing} Y = \{P\}$ and $\delta_P = 1$, then the conductor of Y lies on an arbitrary line passing through P, and vice versa. We give here an example of a singular point P with $\delta_P > 1$ whose conductor lies on a line. Since the problem is local, it is enough to explain it in affine situation. Let $P = (0, 0) \in \mathbb{A}^2$ and $\varepsilon_1(x, y)\, y^2 = \varepsilon_2(x, y)\, x^m$ a local equation of Y near P, where $\varepsilon_1(x, y)$ and $\varepsilon_2(x, y)$ are polynomials with $\varepsilon_1(0, 0)\,\varepsilon_2(0, 0) \neq 0$.

Then $\mathcal{C}_P = (y, x^{\left[\frac{m}{2}\right]})\mathcal{O}_{Y,\,P}$, where $\left[\dfrac{m}{2}\right]$ means the integer part of $\dfrac{m}{2}$.

Hence $\mathcal{C}_P$ lies on the line $y = 0$.

Let $P = (0, 0) \in \mathbb{A}^2$ be a point of a plane curve Y with an affine equation $f(x, y) = 0$. The multiplicity $\mu_P(Y)$ of Y at P is the largest integer m such that $f(x, y) \in (x, y)^m$ in $k[x, y]$.

Lemma 2.9. *If the conductor $\mathcal{C}_P$ at a singular point P of Y lies on a line, then we have $\mu_P(Y) = 2$.*

Proof. We may assume that $P = (0, 0) \in \mathbb{A}^2$ and the conductor $\mathcal{C}_P$ lies on the line $y = 0$. Furthermore, since the valuation $v_{\tilde{P}}$ is upper semi-continuous on the vector space $kx + ky \subset k(X)$ for any $\tilde{P} \in \psi^{-1}(P)$, we may assume that the function $\frac{y}{x}$ belongs in $(\psi_* \mathcal{O}_X)_P \subset k(X)$ after replacing x by a suitable linear form $ax + by$ with $a \neq 0$. Since $y \in \mathcal{C}_P$, we have $\frac{y^2}{x} \in \mathcal{O}_{Y,\,P}$. Hence, there are polynomials $g(x, y)$ and $\varepsilon(x, y)$ with $\varepsilon(0, 0) \neq 0$ such that $\varepsilon(x, y)\, y^2 - xg(x, y) = 0$ on Y. Let Z be the one-dimensional scheme defined by the above equation. Since the coefficient of y^2 in the equation is $\varepsilon(0, 0)$, which is not zero, $\mu_P(Z) \leq 2$. Since $\mu_P(Z) \geq \mu_P(Y)$, we have $\mu_P(Y) = 2$.

3. A Lemma in Projective Geometry

The purpose of this section is to prove the following theorem, which concerns geometry in projective plane. Throughout this section, we fix a projective plane $\mathbb{P}^2$ over an algebraically closed field k. For a 0-dimensional subscheme D of $\mathbb{P}^2$, we denote by $\operatorname{Supp} D$ the set of closed points of D.

Definition 3.1. We say that a 0-dimensional subscheme $D \subset \mathbb{P}^2$ imposes independent conditions on curves of degree ν, if the natural map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\nu)) \to H^0(D, \mathcal{O}_D)$$

is surjective.

Theorem 3.2. *Let n and δ be positive integers with $n \geq 5$, D a 0-dimensional subscheme of $\mathbb{P}^2$ whose length δ is less than or equal to $\min\left\{\frac{n}{2},\, n-4\right\}$. Assume that there is a line $\mathbb{P}^1 \subset \mathbb{P}^2$ such that D is a closed subscheme of the line. Let $P_1, \ldots, P_{n-1}$ are distinct points in $\mathbb{P}^2 \setminus \mathbb{P}^1$. Then the scheme $D + P_1 + \cdots + P_{n-1}$ fails to impose independent conditions on curves of degree $n-3$ if and only if either $P_1, \ldots, P_{n-1}$ are collinear, or there are $n-2$ points of $\{P_1, \ldots, P_{n-1}\}$ such that these points, together with a point of* Supp D, *are collinear.*

The "if" part is easy. Because, in general, if $n-1$ distinct points $R_1, \ldots, R_{n-1}$ are collinear, then the natural map $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-3)) \to \bigoplus_{i=1}^{n-1} k_{R_i}$ is not surjective, where $k_{R_i} (\simeq k)$ is the residue field at R_i.

From now on, we turn to the proof of the "only if" part. When $n = 5$, since the scheme D must be a reduced point by the assumption on δ, the conclusion comes from [2, Appendix A, Exercise 19]. Therefore, we may assume that $n \geq 6$.

Let $\operatorname{Supp} D = \{Q_1, \ldots, Q_m\}$. We want to show that if

(i) $P_1, \ldots, P_{n-1}$ are not collinear, and

(ii) for any $Q_i \in \operatorname{Supp} D$ and any P_j, the $n-1$ points $Q_i, P_1, \ldots, \hat{P}_j, \ldots, P_{n-1}$ are non-collinear, then

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-3)) \to H^0(D, \mathcal{O}_{D+P_1+\cdots+P_{n-1}}) \tag{9}$$

is surjective.

Step 1. In this step, we describe the ring $H^0(\mathcal{O}_{D+P_1+\cdots+P_{n-1}})$ and give a paraphrase of the problem.

Choose a complement of a line $\mathbb{A}^2 \subset \mathbb{P}^2$ with coordinates x, y so that

(i) $\operatorname{Supp} D \cup \{P_1, \ldots, P_{n-1}\} \subset \mathbb{A}^2$,

(ii) the affine line $\mathbb{A}^2 \cap \mathbb{P}^1$ is defined by the equation $y = 0$, where $\mathbb{P}^1$ is the line containing D.

Note that we can use x as a coordinate function of the affine line. Let $x(Q_i) = \alpha_i$ $(i = 1, \ldots, m)$, and $\operatorname{length}_{Q_i} D = \delta_i$. Then we have

$$H^0(\mathcal{O}_D) = k[x] / \prod_{i=1}^{m} (x - \alpha_i)^{\delta_i}.$$

Moreover, let $x(P_j) = \zeta_j$ and $y(P_j) = \eta_j$. Then we have

$$H^0(\mathcal{O}_{D+P_1+\cdots+P_{n-1}})$$

$$= \left(k[x]/\prod_{i=1}^{m}(x-\alpha_i)^{\delta_i}\right) \bigoplus \left(\bigoplus_{j=1}^{n-1} k[x, y]/(x-\zeta_j, y-\eta_j)\right). \tag{10}$$

We denote by $k[x, y]_{\leq n-3}$ the vector space of polynomials in x and y of degree $\leq n-3$. Then homomorphism (9) coincides with the homomorphism

$$k[x, y]_{\leq n-3} \xrightarrow{\Phi} \left(k[x]/\prod_{i=1}^{m}(x-\alpha_i)^{\delta_i}\right) \bigoplus \left(\bigoplus_{j=1}^{n-1} k[x, y]/(x-\zeta_j, y-\eta_j)\right) \tag{11}$$

given by

$$\Phi(h(x, y)) = (h(x, 0); h(\zeta_1, \eta_1), \dots, h(\zeta_{n-1}, \eta_{n-1})). \tag{12}$$

Later on, we will also denote by $h(P_j)$ the value $h(\zeta_j, \eta_j)$ interchangeably.

Let us introduce further notation. For $\gamma = 1, \dots, m$ and $\beta = 0, 1, \dots, \delta_\gamma - 1$, we denote by

$$f_{\gamma\beta} := (x-\alpha_\gamma)^\beta \prod_{\substack{i \text{ with} \\ i \neq \gamma}} (x-\alpha_i)^{\delta_i}.$$

Furthermore, we specify some elements in $H^0(\mathcal{O}_{D+P_1+\cdots+P_{n-1}})$ via the identification (10),

$$\boldsymbol{f}_{\gamma\beta} = (f_{\gamma\beta}; 0, \dots, 0)$$

$$\boldsymbol{e}_j := \left(0; \overset{1}{0}, \dots, \overset{j-1}{0}, \overset{j}{1}, \overset{j+1}{0}, \dots, \overset{n}{0}\right).$$

Then

$$\{\boldsymbol{f}_{\gamma\beta} \mid \gamma = 1, \dots, m, \beta = 0, 1, \dots, \delta_\gamma - 1\} \cup \{\boldsymbol{e}_j \mid j = 1, \dots, n-1\}$$

form k-basis of $H^0(\mathcal{O}_{D+P_1+\cdots+P_{n-1}})$. So, to prove the surjectivity of (9), it suffices to show that each of those vectors is the image of a polynomial in $k[x, y]_{\leq n-3}$ by Φ.

Step 2. We prepare two lemmas in projective geometry. Since both lemmas are elementary, we only formulate them without giving their proof.

Definition 3.3. Four points in $\mathbb{P}^2$ form a *quadrangle* if no 3 of the 4 points is collinear.

Lemma 3.4. *Let S be a subset of $\mathbb{P}^2$ consisting s points with $s \geq 4$. Then no 4 points of S form a quadrangle if and only if S contains $s-1$ collinear points.*

Lemma 3.5. *Let $\{P_1, P_2, P_3, P_4\}$ be a quadrangle in $\mathbb{P}^2$, and S be a finite set of points of $\mathbb{P}^2 \setminus \{P_1, \ldots, P_4\}$. Then there is a quadric form q on $\mathbb{P}^2$ such that $q(P_i) = 0$ for $i = 1, \ldots, 4$ and $q(Q) \neq 0$ for any $Q \in S$.*

Step 3. In this step, we prove that each $\boldsymbol{e}_j$ is the image of a polynomial in $k[x, y]_{\leq n-3}$. Without loss of generality, we may assume that $j = 1$.

Case 3.1. Suppose that the set of $n-2$ points $\{P_2, \ldots, P_{n-2}\}$ contains a quadrangle, say $P_{n-4}, P_{n-3}, P_{n-2}, P_{n-1}$. Then we can find a polynomial q of degree 2 such that $q(P_{n-4}) = \cdots = q(P_{n-1}) = 0$ and $q(P_1) \neq 0$ by Lemma 3.5. For each j with $2 \leq j \leq n-5$, we can choose a polynomial ℓ_j of degree 1 so that $\ell_j(P_j) = 0$ and $\ell_j(P_1) \neq 0$. Then the polynomial $h(x, y) := yq\prod_{j=2}^{n-5}\ell_j$ is of degree $n-3$, and $h(x, y)/h(P_1)$ has the desired property, i.e., $\Phi(h(x, y)/h(P_1)) = \boldsymbol{e}_1$.

Case 3.2. Next suppose that $\{P_2, \ldots, P_{n-1}\}$ does not contain a quadrangle. Then by Lemma 3.4 the set must contain $n-3$ collinear points, say $P_3, \ldots, P_{n-1}$. Let ℓ be a polynomial of degree 1 with $\ell(P_3) = \cdots = \ell(P_{n-1}) = 0$.

Subcase 3.2.1. Suppose that $\ell(P_1) \neq 0$. Choose a linear polynomial ℓ' so that $\ell'(P_2) = 0$ and $\ell'(P_1) \neq 0$. Then the polynomial $y\ell\ell'$ is of degree 3 ($\leq n-3$ because $n \geq 6$), and $\Phi(y\ell\ell') = c\boldsymbol{e}_1$ for a nonzero constant c.

Subcase 3.2.2. Suppose that $\ell(P_1) = 0$. By our two assumptions on the configuration of $\{P_1, \ldots, P_{n-1}\} \cup \operatorname{Supp} D$, the intersection point of the line $\ell = 0$ and $y = 0$ is not a point of $\operatorname{Supp} D$, and $\ell(P_2) \neq 0$. Let ℓ_0 be a linear polynomial which defines the line P_2P_3. Hence $\ell_0(P_1) \neq 0$.

Divide the set of points $\{P_4, P_5, \ldots, P_{3+\delta}\}$ into $m(= {}^{\#}\operatorname{Supp} D)$ subsets as

$$\begin{array}{llll} P_4, & P_5, & \cdots & P_{3+\delta_1} \\ P_{4+\delta_1}, & \cdots & \cdots & P_{3+\delta_1+\delta_2} \\ \vdots & \cdots & \cdots & \vdots \\ P_{4+\delta_1+\cdots+\delta_{m-1}}, & \cdots & \cdots & P_{3+\delta}. \end{array}$$

Note that the set at λ-th row consists of the δ_λ points

$$P_{4+\delta_1+\cdots+\delta_{\lambda-1}}, \ldots, P_{3+\delta_1+\cdots+\delta_\lambda}.$$

Here, recall $\delta_\lambda = \operatorname{length}_{Q_\lambda} D$. For each j with $4 + \delta_1 + \cdots + \delta_{\lambda-1} \leq j \leq 3 + \delta_1 + \cdots + \delta_\lambda$, we consider the polynomial of degree 1

$$\ell_j(x, y) := (x - \alpha_\lambda) - \frac{\zeta_j - \alpha_\lambda}{\eta_j} y,$$

which is an equation of the line P_jQ_λ. Then $\ell_j(x, 0) = (x - \alpha_\lambda)$, $\ell_j(P_j) = 0$ and $\ell_j(P_1) \neq 0$. In particular, $\prod_{j=4}^{3+\delta} \ell_j(x, 0) = \prod_{i=1}^{m} (x - \alpha_i)^{\delta_i}$. For each P_j of the remaining $n - \delta - 4$ points $P_{4+\delta}, \ldots, P_{n-1}$, we choose a polynomial ℓ_j of degree 1 so that $\ell_j(P_j) = 0$ and $\ell_j(P_1) \neq 0$. Then $h := \ell_0 \prod_{j=4}^{n-1} \ell_j$ is of degree $n-3$, and $\Phi\left(\dfrac{h}{h(P_1)}\right) = \boldsymbol{e}_1$.

We devote the remainder of this section to showing the $f_{\gamma\beta}$'s to be in the image of Φ. Without loss of generality, we may assume that $\gamma = 1$.

Step 4. In this step, we explain a technical lemma.

Lemma 3.6. *If we find polynomials* $g_{1\beta}$ *in* x, y *for* $\beta = 0, 1, \ldots, \delta_1 - 1$ *such that* $\deg g_{1\beta} \le n - 3$, $g_{1\beta}(\zeta_i, \eta_i) = 0$ $(i = 1, \ldots, n-1)$, *and*

$$g_{1\beta}(x, 0) = \left(c_\beta^{(\beta)}(x - \alpha_1)^\beta + \sum_{k \ge 1} c_{\beta+k}^{(\beta)}(x - \alpha_1)^{\beta+k}\right)\prod_{j=2}^{m}(x - \alpha_j)^{\delta_j} \tag{13}$$

with a nonzero constant $c_\beta^{(\beta)}$, *then the* $f_{1\beta}$*'s are in the image of* Φ.

Proof. Since $\Phi(g_{1\beta}) = (g_{1\beta}(x, 0); 0, \ldots, 0)$ and (13), we have

$$\begin{pmatrix} \Phi(g_{10}) \\ \vdots \\ \Phi(g_{1,\delta_1-1}) \end{pmatrix} = \begin{pmatrix} c_0^{(0)} & \cdots & c_{\delta_1-1}^{(0)} \\ & \ddots & \vdots \\ 0 & & c_{\delta_1-1}^{(\delta_1-1)} \end{pmatrix} \begin{pmatrix} f_{10} \\ \vdots \\ f_{1,\delta_1-1} \end{pmatrix}$$

in $\left(k[x]/\prod_{i=1}^{m}(x - \alpha_i)^{\delta_i}\right) \bigoplus \left(\bigoplus_{j=1}^{n-1} k[x, y]/(x - \zeta_j, y - \eta_j)\right)$.

Step 5. Let $\{\eta^{(1)}, \ldots, \eta^{(e)}\} \subset k^\times$ be the set of possible values of the y-coordinates of $P_1, \ldots, P_{n-1}$, that is, e is the number of different values among $\eta_1, \ldots, \eta_{n-1}$. In this step, we consider the case $e \le n - 2 - \delta$. Let $h(x, y) := f_{1\beta}(x)\prod_{i=1}^{e}(y - \eta^{(i)})$. Then $\deg h \le \delta - 1 + e \le n - 3$ by the assumption, and $\Phi(h) = (h(x, 0); h(P_1), \ldots, h(P_{n-1})) = cf_{1\beta}$ for some nonzero constant c.

Step 6. We handle the case $e \ge n - 1 - \delta$ by dividing it into several subcases. Note that $e \ge \delta - 1$ because $n \ge 2\delta$ by an original assumption on δ. Without loss of generality, we may assume that

$$\eta^{(i)} = y(P_{n-i}) \quad (i = 1, \ldots, e).$$

Let

$$\widetilde{f}_{1\beta}(x, y) = f_{1\beta}(x) - (c_1 y + c_2 y^2 + \cdots + c_{\delta-1} y^{\delta-1}),$$

where $(c_1, \ldots, c_{\delta-1})$ is the solution of

$$\begin{pmatrix} \eta^{(1)} & \eta^{(1)^2} & \cdots & \eta^{(1)^{\delta-1}} \\ \eta^{(2)} & \eta^{(2)^2} & \cdots & \eta^{(2)^{\delta-1}} \\ \vdots & \vdots & & \vdots \\ \eta^{(\delta-1)} & \eta^{(\delta-1)^2} & \cdots & \eta^{(\delta-1)^{\delta-1}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{\delta-1} \end{pmatrix} = \begin{pmatrix} f_{1\beta}(P_{n-1}) \\ f_{1\beta}(P_{n-2}) \\ \vdots \\ f_{1\beta}(P_{n-\delta+1}) \end{pmatrix}.$$

Then we have that

$$\deg \widetilde{f}_{1\beta}(x, y) = \delta - 1, \tag{14}$$

$$\widetilde{f}_{1\beta}(x, 0) = f_{1\beta}(x), \tag{15}$$

and

$$\widetilde{f}_{1\beta}(P_{n-i}) = 0 \quad (i = 1, \ldots, \delta - 1). \tag{16}$$

Now, we look at the configuration of the remaining $n - \delta$ points $P_1, \ldots, P_{n-\delta}$.

Case 6.1. Suppose that the set $\{P_1, \ldots, P_{n-\delta}\}$ contains a quadrangle. We may assume that P_1, P_2, P_3, P_4 form a quadrangle. Choose a quadric polynomial q so that $q(P_1) = \cdots = q(P_4) = 0$ and $q(Q_1) \neq 0$. Furthermore, for each P_i with $5 \leq i \leq n - \delta$, choose a polynomial ℓ_i of degree 1 so that $\ell_i(P_i) = 0$ and $\ell_i(Q_1) \neq 0$. Then the polynomial $h := \widetilde{f}_{1\beta} q \prod_{i=5}^{n-\delta} \ell_i$ is of degree $n - 3$ because of (14), and $h(P_1) = \cdots = h(P_{n-1}) = 0$ because of (16). Since $q(Q_1) \prod_{i=5}^{n-\delta} \ell_i(Q_1) \neq 0$ and (15), the polynomial $h(x, 0)$ is of the form (13) in Lemma 3.6. So this case has been settled.

Case 6.2. Suppose that the set $\{P_1, \ldots, P_{n-\delta}\}$ does not contain any quadrangles. Since $n - \delta - 1$ points of the set are collinear by Lemma 3.4, we may assume that $P_1, \ldots, P_{n-\delta-1}$ are collinear. We denote by L the line containing those $n - \delta - 1$ points.

Subcase 6.2.1. Suppose that L does not contain Q_1. Let $\ell = 0$ be an equation of L, and $\ell_{n-\delta}$ a polynomial of degree 1 with $\ell_{n-\delta}(P_{n-\delta}) = 0$ and $\ell_{n-\delta}(Q_1) \neq 0$. From (14) and the original assumption $\delta \leq n - 4$, $\deg \widetilde{f}_{1\beta}\ell\ell_{n-\delta} = \delta + 1 \leq n - 3$. Because of our choice of the polynomials, $\widetilde{f}_{1\beta}\ell\ell_{n-\delta}$ has the desired properties described in Lemma 3.6.

Subcase 6.2.2. Finally suppose that $Q_1 \in L$. Since $P_1, \ldots, P_{n-\delta-1}$, $Q_1 \in L$ and the line L is not parallel to the line $y = 0$ in the $\mathbb{A}^2$, the y-coordinates of these points are mutually different. In particular, since $n - \delta - 1 \geq \delta - 1$ by one of our original assumptions, the y-coordinates of $P_1, \ldots, P_{\delta-1}$ are different.

Now, we rechoose the coefficients $c_1, \ldots, c_{\delta-1}$ of $\widetilde{f}_{1,\beta}$ by the solutions of

$$(y(P_i)^j)_{\substack{1\leq i\leq\delta-1\\ 1\leq j\leq\delta-1}}\begin{pmatrix}\vdots\\ c_j\\ \vdots\end{pmatrix}_j = \begin{pmatrix}\vdots\\ f_{1\beta}(P_i)\\ \vdots\end{pmatrix}_i.$$

Then we have that (14), (15), and

$$\widetilde{f}_{1\beta}(P_i) = 0 \quad (i = 1, \ldots, \delta - 1).$$

For the configuration of the remaining $n - \delta$ points $P_\delta, P_{\delta+1}, \ldots, P_{n-1}$, there are three possibilities:

(i) the set $\{P_\delta, \ldots, P_{n-1}\}$ contains a quadrangle;

(ii) there is a line M on which $n - \delta - 1$ of the $n - \delta$ points lie, but Q_1 does not lie on the line M;

(iii) there is a line M which $n - \delta - 1$ of the $n - \delta$ points and Q_1 lie on.

If the first or the second case happens, we can find a desired polynomial by the same argument to Case 6.1 or Subcase 6.2.1 respectively. So we have to consider the third case, and treat the case in the next step.

Step 7. First we show that the line L, which $P_1, \ldots, P_{n-\delta-1}$ lie on, and the line M, which $n - \delta - 1$ of the $n - \delta$ points $P_\delta, \ldots, P_{n-1}$ lie on, meet exactly at Q_1. Since $\delta \le (n - \delta - 1) + 1$, the union $\{P_1, \ldots, P_{n-\delta-1}\}$ with $\{P_\delta, \ldots, P_{n-1}\}$ covers $\{P_1, \ldots, P_{n-1}\}$. Hence, if $L = M$, then the line contains at least $n - 2$ points of $\{P_1, \ldots, P_{n-1}\}$, which contradicts with the original assumption because Q_1 is on the line.

In particular, $\{P_1, \ldots, P_{n-\delta-1}\} \cap \{P_\delta, \ldots, P_{n-1}\}$ is empty or consists of one point. Since

$$\#\{P_1, \ldots, P_{n-1}\} = (n - \delta - 1) + (n - \delta) - \#(\{P_1, \ldots, P_{n-\delta-1}\} \cap \{P_\delta, \ldots, P_{n-1}\}),$$

we have $n = 2\delta$ or $2\delta + 1$. We consider the two cases separately.

Case 7.1. If $n = 2\delta + 1$, then

$$L \ni P_1, \ldots, P_{n-\delta-1} = P_\delta$$
$$M \ni P_{\delta+1}, \ldots, P_{n-1}.$$

Note that neither of the two lines L and M is parallel to the line $y = 0$. Let $\ell_1 = 0$ and $\ell_2 = 0$ be equations of L and M respectively. We choose $c_1, \ldots, c_\delta \in k$ as the solutions of

$$(y(P_i)^j)_{\substack{1 \le i \le \delta \\ 1 \le j \le \delta}} \begin{pmatrix} \vdots \\ c_j \\ \vdots \end{pmatrix}_j = \begin{pmatrix} \vdots \\ f_{1\beta}(P_i)/\ell_2(P_i) \\ \vdots \end{pmatrix}_i$$

and $d_1, \ldots, d_\delta \in k$ as those of

$$(y(P_{\delta+i})^j)_{\substack{1\le i\le\delta \\ 1\le j\le\delta}} \begin{pmatrix} \vdots \\ d_j \\ \vdots \end{pmatrix}_j = \begin{pmatrix} \vdots \\ f_{1\beta}(P_{\delta+i})/\ell_1(P_{\delta+i}) \\ \vdots \end{pmatrix}_i .$$

Let

$$h(x,\ y) := f_{1\beta} - (\ell_2(c_1 y + \cdots + c_\delta y^\delta) + \ell_1(d_1 y + \cdots + d_\delta y^\delta)).$$

Then we have $h(x,\ 0) = f_{1\beta}$, $h(P_i) = 0$ $(i = 1, \ldots, n-1)$, and $\deg h \le \delta + 1 \le n - 3$ because $\delta \le n - 4$. Therefore, we have $\Phi(h) = f_{1\beta}$.

Case 7.2. Lastly we consider the case $n = 2\delta$. We may assume that $L \ni P_1, \ldots, P_{\delta-1}$ and $M \ni P_{\delta+1}, \ldots, P_{2\delta-1}$. (We don't know where P_δ is.) We choose the constants $c_1, \ldots, c_{\delta-1}$, $d_1, \ldots, d_{\delta-1}$ by the same way with the previous case. Moreover, we choose a polynomial ℓ_δ of degree 1 so that $\ell_\delta(P_\delta) = 0$ and $\ell_\delta(Q_1) \neq 0$. Then the polynomial

$$h(x,\ y) := (f_{1\beta} - (\ell_2(c_1 y + \cdots + c_{\delta-1} y^{\delta-1}) + \ell_1(d_1 y + \cdots + d_{\delta-1} y^{\delta-1})))\,\ell_\delta$$

is of degree $\le \delta + 1\,(\le n-3)$, and $\Phi(h) = cf_{1\beta}$ for a nonzero constant c.

We have completed the proof of Theorem 3.2.

4. Plane Curves with Aligned Conductors

Now, we go back to the circumstances in Section 2. The aim of this section is to study the linear system g_d^2 cut out on X by lines for a singular plane curve Y of degree d whose conductors is aligned. As is the same usage in Section 2, δ denotes the length of the scheme of conductors $D = \operatorname{Spec} \mathcal{O}_Y/\mathcal{C}$.

Lemma 4.1. *If the conductors of Y is aligned, then we have $2\delta \le d$.*

Proof. From Lemma 2.5, together with Lemma 2.3, we have $h^0(X,\ \psi^*\mathcal{O}_Y(1)(-C)) > 0$. Hence $\deg \psi^*\mathcal{O}_Y(1)(-C) \ge 0$, which means $d - 2\delta \ge 0$.

Theorem 4.2. *Assume that the conductors of Y is aligned.*

(a) *If either* $d \geq 5$, *or* $d = 4$ *and* $\delta = 1$, *then the linear system* $\psi^*|\mathcal{O}_Y(1)|$ *is complete.*

(b) *If either* $d \geq 7$, *or* $d = 6$ *and* $\delta \leq 2$, *or* $d = 5$ *and* $\delta = 1$, *then the complete linear system* $\psi^*|\mathcal{O}_Y(1)|$ *is the unique* g_d^2 *on* X.

Proof. We denote by $\mathbb{P}^1$ the line containing the scheme of conductors D of Y, and by ρ the closed immersion $D \hookrightarrow \mathbb{P}^1$. (When Sing Y consists of one point P with $\delta_P = 1$, we fix an arbitrary line $\mathbb{P}^1$ passing through P.)

(a) We already established the completeness of the linear system at Lemma 2.6 if the characteristic of k is not 2, however, here we give a more conceptual proof of the completeness, which works in an arbitrary characteristic.

By Theorem A.4 in Appendix, it suffices to show that

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-4)) \to H^0(D, \mathcal{O}_D)$$

is surjective. It is obvious that the surjectivity is equivalent to that of

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-4)) \xrightarrow{\rho^*} H^0(D, \mathcal{O}_D).$$

Since length $D = \delta$, we have $\mathcal{I}_{D,\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-\delta)$. Hence the morphism ρ^* fits in the exact sequence

$$H^0(\mathcal{O}_{\mathbb{P}^1}(d-4)) \xrightarrow{\rho^*} H^0(\mathcal{O}_D) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-4-\delta)) \to 0.$$

Therefore, the morphism ρ^* is surjective if and only if $\delta \leq d-3$. By Lemma 4.1, the condition is equivalent to either $d \geq 5$ or $d = 4$ and $\delta = 1$ under the assumption $d \geq 4$.

(b) Let g_d^2 be a linear system on X of degree d and of projective dimension 2. We have to show that $g_d^2 = \psi^*|\mathcal{O}_Y(1)|$.

The first claim is that even if the g_d^2 has base points, we may assume that

(i) the image of each base point by ψ not to lie on the line $\mathbb{P}^1$, and

(ii) the divisor of base points of g_d^2 to have no multiple points.

To show the claim, let B be the divisor of base points of g_d^2, and $b := \deg B$. Choose general points $Q_1, \ldots, Q_b \in X$ and consider the linear system $g_d^2(-B) + Q_1 + \cdots + Q_b$, which is also a linear system of degree d and of projective dimension 2. Moreover, the linear system satisfies conditions (i) and (ii). If we can prove that $g_d^2(-B) + Q_1 + \cdots + Q_b = \psi^*|\mathcal{O}_Y(1)|$, then the linear system is free from base points, that is, $b = 0$. So g_d^2 itself coincides with $\psi^*|\mathcal{O}_Y(1)|$.

By the above assumption, we can choose a general member $P_1 + \cdots + P_d \in g_d^2$ such that $P_1, \ldots, P_d$ are distinct points of $X \setminus \psi^{-1}(\mathbb{P}^1)$. Since the linear system is of projective dimension 2, there are two points in the d points, say P_1, P_2, such that

$$\dim|P_3 + \cdots + P_d| = 0. \tag{17}$$

In particular,

$$\dim|P_2 + \cdots + P_d| = \dim|P_1 + P_3 + \cdots + P_d| = 1. \tag{18}$$

Note that we can regard each P_i as a point of $Y \setminus \mathbb{P}^1$.

The second claim is that the natural map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(d-3)) \xrightarrow{\rho_i^*} H^0(\mathcal{O}_{D+P_i+P_3+\cdots+P_d})$$

is not surjective for $i = 1$ and 2. To show the claim, without loss of generality, we may assume that $i = 2$. Suppose that ρ_2^* was surjective. Then, by observing the commutative diagram

$$\begin{array}{ccccccc} & & & & H^0(\mathcal{O}_{\mathbb{P}^2}(d-3)) & & \\ & & & & \downarrow \simeq & \searrow \rho_2^* & \\ 0 \to H^0\left(\mathcal{O}_Y(d-3)\left(-D-\sum_{i=2}^{d} P_i\right)\right) & \to & & & H^0(\mathcal{O}_Y(d-3)) & \to & H^0(\mathcal{O}_{D+P_2+\cdots+P_d}) \end{array}$$

with exact row, we would have

$$h^0\left(\mathcal{O}_Y(d-3)\left(-D-\sum_{i=2}^{d} P_i\right)\right) = g-d+1.$$

On the other hand, since

$$H^0(Y, \mathcal{O}_Y(d-3)(-D)) \simeq H^0(X, (\psi^*\mathcal{O}_Y(d-3))(-C))$$

by Lemma 2.3 and $P_1, \ldots, P_d \in Y \setminus \operatorname{Sing} Y \subset X$, we have

$$h^0\left(Y, \mathcal{O}_Y(d-3)\left(-D-\sum_{i=2}^{d} P_i\right)\right) = h^0\left(X, (\psi^*\mathcal{O}_Y(d-3))\left(-C-\sum_{i=2}^{d} P_i\right)\right)$$

because the diagram

$$\begin{array}{ccc} H^0(Y, \mathcal{O}_Y(d-3)(-D)) & \xrightarrow{\sim} & H^0(X, (\psi^*\mathcal{O}_Y(d-3))(-C)) \\ & \searrow \quad \swarrow & \\ & H^0(\mathcal{O}_{P_2+\cdots+P_d}) & \end{array}$$

is commutative. Since $(\psi^*\mathcal{O}_Y(d-3))(-C) \simeq \omega_X$, we have

$$h^0\left(X, (\psi^*\mathcal{O}_Y(d-3))\left(-C-\sum_{i=2}^{d} P_i\right)\right) = g-d+h^0\left(\mathcal{O}_X\left(\sum_{i=2}^{d} P_i\right)\right)$$

(by Riemann-Roch)

$$= g-d+2 \qquad \text{(by (18))}.$$

Hence the supposed surjectivity of ρ^* contradicts with the correct dimension of

$$H^0\left(X, (\psi^*\mathcal{O}_Y(d-3))\left(-C-\sum_{i=2}^{d} P_i\right)\right).$$

Therefore, considering Lemma 4.1, we can apply Theorem 3.2 for D and the $d-1$ points $P_2, P_3, \ldots, P_d$, and we know that

(1a) $P_2, \ldots, P_d$ are collinear, or

(1b) there is a point $Q_i \in \operatorname{Supp} D$ such that Q_i and some $d-2$ points of $\{P_2, \ldots, P_d\}$ are collinear.

Also for $P_1, P_3, \ldots, P_d$, we have

(2a) $P_1, P_3, \ldots, P_d$ are collinear, or

(2b) there is a point $Q_j \in \operatorname{Supp} D$ such that Q_j and some $d-2$ points of $\{P_1, P_3, \ldots, P_d\}$ are collinear.

Hence, logically, there are the four possibilities of the circumstances

Case 1. (1a) and (2a) occur,

Case 2. (1a) and (2b) occur,

Case 3. (1b) and (2a) occur,

Case 4. (1b) and (2b) occur.

If (1a) and (2b) occur at the same time, then the line determined by (1a) coincides with that by (2b) because $d \geq 5$. Let L be the line. We denote by $(Y.L)$ the intersection number of Y and L, and by $i(Y.L;\ P)$ the intersection multiplicity of Y and L at P. Then we have

$$d = (Y.L) \geq \sum_{i=2}^{d} i(Y.L;\ P_i) + i(Y.L;\ Q_i)$$

$$\geq (d-1) + 2,$$

which is a contradiction. So Case 2 is not the case, and neither is Case 3. Next, we consider Case 4. Let L_1 and L_2 be lines determined by the conditions (1b) and (2b) respectively. When $d \geq 6$, since L_1 and L_2 have at least two common points, the two lines must coincide and $Q_i = Q_j$. When $d = 5$, the set $L_1 \cap L_2 \cap \{P_1, \ldots, P_5\}$ contains at least one point. Since we assumed $\delta = 1$ for $d = 5$, the equality $Q_i = Q_j$ is automatically satisfied. Hence $L_1 = L_2$. If $d - 1$ points of $\{P_1, \ldots, P_d\}$ lie on the line, we come to a contradiction by the same argument in Case 2. If the line, say L, contains exactly $d - 2$ points of $\{P_1, \ldots, P_d\}$, these points must be $\{P_3, \ldots, P_d\}$. Since Q_i is a singular point of a degree d curve Y, we have

$$Y \cap L = Q_i + P_3 + \cdots + P_d \quad \text{and} \quad i(Y.L; Q_i) = 2.$$

Hence, the linear system $|P_3 + \cdots + P_d|$ on X contains the g^1_{d-2} cut out by lines through Q_i, which contradicts with (17). Therefore, the only possibility is Case 1, which means $P_1, \ldots, P_d$ are collinear.

5. Supplement

In Theorem 4.2, we excluded a few cases from consideration. Taking account of Lemma 4.1, the remaining cases for the completeness of $\psi^*|\mathcal{O}_Y(1)|$ are "$d = 4$ with $\delta = 2$" and "$d = 3$ with $\delta = 1$," and for the uniqueness of g^2_d "$d = 6$ with $\delta = 3$," "$d = 5$ with $\delta = 2$," "$d = 4$ with $\delta = 2$," "$d = 4$ with $\delta = 1$" and "$d = 3$ with $\delta = 1$."

Here we discuss both problems for those cases.

The case "$d = 3$ with $\delta = 1$"

In this case, $g = 0$. Hence any g^2_3 is not complete, and the g^2_3's form a one-dimensional family.

The case "$d = 4$ with $\delta = 2$"

In this case, $g = 1$. Hence any g^2_4 is not complete, and the g^2_4's form a two-dimensional family.

The case "$d = 4$ with $\delta = 1$"

In this case, $g = 2$. By the Riemann-Roch theorem, any g_4^2 on the curve X is complete. Since any invertible sheaf M of degree 4 with $h^0(M) = 3$ on X can be written as $M \simeq \omega_X(P + Q)$ for some $P, Q \in X$, the g_4^2's form a two-dimensional family.

The case "$d = 5$ with $\delta = 2$"

The curve X is of genus 4. The completeness of $\psi^*|\mathcal{O}_Y(1)|$ is guaranteed by Theorem 4.2(a), though we can know it from another reason. Actually, any g_5^2 on X must be complete because of Clifford's theorem, and the family of the g_5^2's coincides with $\{|K(-P)| \mid P \in X\}$ by the Riemann-Roch theorem, where K is a canonical divisor.

The case "$d = 6$ with $\delta = 3$"

The linear system $\psi^*|\mathcal{O}_Y(1)|$ is complete by Theorem 4.2(a). For the uniqueness, we need a little more argument.

Proposition 5.1. *If $d = 6$ and $\delta = 3$, then the complete linear system $\psi^*|\mathcal{O}_Y(1)|$ is the unique g_6^2.*

Proof. Note that the genus g of the curve X is 7. First we show that X is neither hyperelliptic, trigonal nor elliptic-hyperelliptic. If we fix a non-singular point on Y, we can get a pencil g_5^1 on X cut out by lines through the point. Since the existence of g_2^1 is incompatible with that of g_5^1 because of the inequality of Castelnuovo-Severi (see, [1, Theorem 3.5] or [2, VIII, Exercises C-1]), X is nonhyperelliptic. If we consider the pencil on X cut out by lines through an assigned singular point of Y, then we have g_4^1 because the multiplicity of Y at the point is 2 by Lemma 2.9. Hence X is not trigonal because of the inequality of Castelnuovo-Severi. By the same reason, we know that there are no morphisms of degree 2 from the curve X to an elliptic curve because of the existence of g_5^1.

Next, we show that any g_6^2 is free from base points, and the morphism associated to the linear system is birational onto its image. Let B be the set of base points of the g_6^2. If deg B was greater than 1, X would be trigonal or hyperelliptic. So $\deg B \leq 1$. If $\deg B = 1$, we have a linear system g_5^2, and the linear system is very ample by the same reason. Hence, the genus of X must be 6, which is a contradiction. Therefore, the g_6^2 has no base points, and we have a morphism $\phi_{g_6^2} : X \to \mathbb{P}^2$. If the morphism was not birational onto $\phi_{g_6^2}(X)$, the curve X would be either elliptic-hyperelliptic or trigonal.

Let $Q_1 + \cdots + Q_6$ be a general member of the g_6^2, and $M = \mathcal{O}_X(Q_1 + \cdots + Q_6)$. We may assume that $g_6^2(-Q_1)$ and $g_6^2(-Q_2)$ are free from base points because $\phi_{g_6^2}$ is birational. Let V be the 2-dimensional subspace of $H^0(X, M(-Q_1))$ corresponding to the linear system $g_6^2(-Q_1)$. (Note that we have not yet known the completeness of the g_6^2.) Let us consider the natural map

$$H^0(X, \psi^*(\mathcal{O}_Y(1))) \otimes V \xrightarrow{\mu} H^0(X, \psi^*(\mathcal{O}_Y(1)) \otimes M(-Q_1)).$$

Case 1. $\operatorname{Ker} \mu = (0)$.

In this case, since

$$h^0(X, \psi^*(\mathcal{O}_Y(1)) \otimes M(-Q_1)) \geq 6(= g-1)$$

and

$$\deg \psi^*(\mathcal{O}_Y(1)) \otimes M(-Q_1) = 11(= 2g-3),$$

there is a point $R \in X$ such that

$$\psi^*(\mathcal{O}_Y(1)) \otimes M(-Q_1) \simeq \omega_X(-R) \simeq \psi^*(\mathcal{O}_Y(3))(-C-R).$$

On the other hand, $\psi^*(\mathcal{O}_Y(1))(-C) \simeq \mathcal{O}_X$, since $h^0(\psi^*(\mathcal{O}_Y(1))(-C)) > 0$ by Lemma 2.5 and $\deg \psi^*(\mathcal{O}_Y(1))(-C) = 0$ by our assumption. Hence we have

$$\psi^*(\mathcal{O}_Y(1))(Q_1) \simeq M(R),$$

which means that

$$Q_2 + \cdots + Q_6 + R \in |\psi^*(\mathcal{O}_Y(1))|,$$

in particular $\psi(Q_2), \ldots, \psi(Q_6)$ are collinear.

Case 2. $\operatorname{Ker} \mu \neq (0)$.

Since

$$\operatorname{Ker} \mu \simeq H^0\left(X, \psi^*(\mathcal{O}_Y(1))\left(-\sum_{i=2}^{6} Q_i\right)\right)$$

by the base-point-free pencil trick, the non-triviality of $\operatorname{Ker} \mu$ implies that there is a point $R \in X$ such that $Q_2 + \cdots + Q_6 + R \in |\psi^*(\mathcal{O}_Y(1))|$.

Therefore, $\psi(Q_2), \ldots, \psi(Q_6)$ are collinear in either case occurs. The same thing is true for $Q_1, Q_3, \ldots, Q_6$. So we have $\psi(Q_1), \ldots, \psi(Q_6)$ are collinear, which means $\mathcal{O}_X(Q_1 + \cdots + Q_6) \simeq \psi^*(\mathcal{O}_Y(1))$.

Appendix

Throughout this appendix, Y denotes an irreducible curve of degree $d > 1$ in $\mathbb{P}^2$ with $\operatorname{Sing} Y \neq \varnothing$, and $X \overset{\psi}{\to} Y$ the normalization of Y. We do not assume the conductors of Y to be aligned. The purpose of the appendix is to explain a necessary and sufficient condition for the linear system $\psi^*|\mathcal{O}_Y(1)|$ to be complete.

Lemma A.1. *Let Z be a finite subscheme of Y. Then*

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n) \otimes \mathcal{I}_Z) \to H^0(Y, \mathcal{O}_Y(n)(-Z))$$

is surjective for any positive integer n. Moreover, if $n < d$, *then the morphism is an isomorphism.*

Proof. Let us consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^0(\mathcal{O}_{\mathbb{P}^2}(n)\otimes\mathcal{I}_Z) & \xrightarrow{u} & H^0(\mathcal{O}_Y(n)(-Z)) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathcal{O}_{\mathbb{P}^2}(n-d)) & \to & H^0(\mathcal{O}_{\mathbb{P}^2}(n)) & \to & H^0(\mathcal{O}_Y(n)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & H^0(\mathcal{O}_Z) & = & H^0(\mathcal{O}_Z) & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where all rows and columns are exact. Hence, by the snake lemma, the morphism u is surjective. Moreover, since $\operatorname{Ker} u \simeq H^0(\mathcal{O}_{\mathbb{P}^2}(n-d))$, u is an isomorphism if $n < d$.

Corollary A.2. *For the scheme of conductors D of Y, we have*

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n) \otimes \mathcal{I}_D) \simeq H^0(Y, \mathcal{O}_Y(n)(-D)) \simeq H^0(X, (\psi^*\mathcal{O}_Y(n))(-C))$$

for n with $1 \le n < d$.

Proof. This is just a combination of Lemmas 2.3 and A.1.

Remark A.3. The scheme of conductors D of Y imposes independent conditions on curves of degree $d-3$. In fact, by Corollary A.2,

$$h^0(\mathcal{O}_{\mathbb{P}^2}(d-3)\otimes\mathcal{I}_D)=h^0(X,\omega_X)$$

$$=p_a(Y)-\delta$$

$$=h^0(\mathcal{O}_{\mathbb{P}^2}(d-3))-h^0(D,\mathcal{O}_D).$$

Hence the natural map

$$H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(d-3))\to H^0(D,\mathcal{O}_D)$$

is surjective.

We have come to the goal of the appendix.

Theorem A.4. *The linear system g_d^2 cut out on X by lines, that is to say the linear system $\psi^*|\mathcal{O}_Y(1)|$, is complete if and only if the scheme of conductors D of Y imposes independent conditions on curves of degree $d-4$.*

Proof. Our linear system g_d^2 on X corresponds to the image of the natural map

$$H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1))\xrightarrow{\sim}H^0(Y,\mathcal{O}_Y(1))\hookrightarrow H^0(X,\psi^*\mathcal{O}_Y(1)).$$

Hence, the g_d^2 is complete if and only if $h^0(X,\psi^*\mathcal{O}_Y(1))=3$, which is equivalent to the condition

$$h^1(X,\psi^*\mathcal{O}_Y(1))=g-d+2$$

by the Riemann-Roch theorem. From the Serre duality, together with (7), we have

$$h^1(X,\psi^*\mathcal{O}_Y(1))=h^0(X,(\psi^*\mathcal{O}_Y(d-4))(-C)),$$

and from Corollary A.2,

$$h^0(X, (\psi^*\mathcal{O}_Y(d-4))(-C)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-4) \otimes \mathcal{I}_D).$$

On the other hand, we have

$$\begin{aligned} h^0(\mathcal{O}_{\mathbb{P}^2}(d-4)) - h^0(D, \mathcal{O}_D) &= \frac{1}{2}(d-2)(d-3) - \delta \\ &= \frac{1}{2}(d-1)(d-2) - \delta - d + 2 \\ &= g - d + 2. \end{aligned}$$

To sum up: we have that the g_d^2 is complete if and only if the natural map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-4)) \to H^0(D, \mathcal{O}_D)$$

is surjective.

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References

[1] R. D. M. Accola, Topics in the theory of Riemann surfaces, Lecture Notes in Math. 1595, Springer-Verlag, Berlin, Heidelberg, 1994.

[2] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, Geometry of Algebraic Curves, Vol. 1, Grundlehren Math. Wiss. 267, Springer-Verlag, New York, 1985.

[3] V. Bayer and A. Hefez, Strange curves, Comm. Algebra 19 (1991), 3041-3059.

[4] M. Coppens and T. Kato, The gonality of smooth curves with plane models, Manuscripta Math. 70 (1990), 5-25.

[5] D. Eisenbud, M. Green and J. Harris, Cayley-Bacharach theorems and conjectures, Bull. Amer. Math. Soc. 33 (1996), 295-324.

[6] D. Gorenstein, An arithmetic theory of adjoint plane curves, Trans. Amer. Math. Soc. 72 (1952), 414-436.

[7] L. Gruson, R. Lazarsfeld and C. Peskine, On a theorem of Castelnuovo, and the equations defining space curves, Invent. Math. 72 (1983), 491-506.

[8] R. Hartshorne, Algebraic Geometry, Graduate Texts in Math. 52, Springer-Verlag, New York, 1977.

[9] R. Hartshorne, Generalized divisor on Gorenstein curves and a theorem of Noether, J. Math. Kyoto Univ. 26 (1986), 375-386.

[10] J.-P. Serre, Groupes Algébriques et Corps de Classes, Hermann, Paris, 1959.

[11] L. Szpiro, Lectures on equations defining space curves, Tata Inst. Fund. Res. (Math.) 62, Springer-Verlag, Heidelberg, 1979.

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