SEPARABLE GONALITY OF A GORENSTEIN CURVE

Masaaki Homma

0. Introduction

Just as in the case of smooth curves, an integral projective curve X, which may have singular points, of arithmetic genus $g \geq 2$ is said to be hyperelliptic if there is a finite morphism $X \to \mathbf{P}^1$ of degree 2. However, there is a phenomenon which never happens in the case of smooth hyperelliptic curves; that is, the degree-two morphism may be inseparable. A hyperelliptic curve with this property is said to be of *inseparable type*. The complete picture of singular hyperelliptic curves can be found in [3].

On the other hand, any (singular or nonsingular) hyperelliptic curve is Gorenstein ([4, Th. 15], [3, (2.2)]), i.e. the dualizing sheaf of the curve is invertible. The purpose of this short note is to give a characterization of hyperelliptic curves of inseparable type in the category of Gorenstein curves in terms of the *separable gonality* of a curve, which is defined to be the smallest possible degree of a finite separable morphism from the curve to the projective line. Our result should be placed in a more general context of a divisor theory of a Gorenstein curve; however, here we will give a makeshift proof to it.

Throughout this note, we will assume the ground field K to be algebraically closed.

1. The inequality $k_s \leq g + 1$

Let X be an integral projective curve of genus g. We will show that *there is a finite separable morphism $X \to \mathbf{P}^1$ of degree less than or equal to $g + 1$.*

In fact, let us take pairwise distinct $g+1$ smooth points $P_1, \cdots, P_{g+1}$ of X. Then we have $h^0(\mathcal{O}_X(P_1 + \ldots + P_{g+1})) \geq 2$ by the Riemann-Roch theorem. Hence there is a non-constant function $f : X \to \mathbf{P}^1$ whose pole divisor $(f)_\infty$ is at most $P_1 + \cdots + P_{g+1}$. Since $v_P(f) = -1$ for any $P \in (f)_\infty$, the morphism f is separable.

□

Therefore we can define the *separable gonality* $k_s = k_s(X)$ of X in the way which was mentioned in Introduction.

2. Main result

Our theorem is as follows.

Theorem. *Let X be a Gorenstein curve of genus $g \geq 2$. Then*

$$k_s(X) \leq g+1.$$

Furthermore, equality occurs if and only if X is hyperelliptic of inseparable type.

Proof. The first part of the assertion has been proved in the previous section.

First we will show that

$$k_s(X) \geq g+1$$

for a hyperelliptic curve X of inseparable type. By definition, there are two particular finite morphisms from X to $\mathbf{P}^1$; one, say x, is an inseparable morphism of degree 2 and the other, say y, is separable of degree k_s. Since the function field of X is $K(x,y)$, the morphism

$$(x,y) : X \to \mathbf{P}^1 \times \mathbf{P}^1$$

is birational onto its image. Hence we have

$$g \leq (2-1)(k_s - 1)$$

by virtue of Castelnuovo's inequality.

Next we will show that

$$k_s(X) \leq g$$

if X is a Gorenstein curve that is hyperelliptic of separable type (i.e. the degree-two morphism is separable) or nonhyperelliptic. By definition, $k_s(X) = 2$ if X is hyperelliptic of separable type. Let X be a nonhyperelliptic Gorenstein curve. Then the canonical linear system is very ample ([4, Th. 17], 2, (1.6)], [5, (3.3)]), that is, X can be embedded in $\mathbf{P}^{g-1}$ as a curve of degree $2g-2$. Hence, by using Bertini's theorem [1, II (8.18) and (8.18.1)], we can find pairwise distinct $2g-2$ smooth points $P_1, \cdots, P_{2g-2}$ so that

$$h^0(\mathcal{O}_X(P_1 + \ldots + P_{2g-2})) = g\,.$$

Hence $h^0(\mathcal{O}_X(P_1 + \ldots + P_g)) \geq 2$. Therefore we can conclude that $k_s \leq g$ by the same argument in Section 1.

□

Remark. The first part of the statement of Theorem holds without assuming X to be Gorenstein, but the second part does not.

In fact, let us consider the curve Y_g obtained from the projective line $\mathbf{P}^1$ by replacing the local ring $\mathcal{O}_{\mathbf{P}^1,0}$ by

$$K + t^{g+1}\mathcal{O}_{\mathbf{P}^1,0}\,,$$

where t is a uniformizing function on $\mathbf{P}^1$ so that $t\mathcal{O}_{\mathbf{P}^1,0}$ is the maximal ideal of $\mathcal{O}_{\mathbf{P}^1,0}$. If $g \geq 2$, then Y_g is a non-Gorenstein curve of genus g. Looking at the local ring of Y_g at 0, we know that every nonconstant function $Y_g \to \mathbf{P}^1$ is of degree greater than g. Hence Y_g is a nonhyperelliptic curve with $k_s(Y_g) = g+1$.

Acknowledgment. This work was done while the author was visiting IMPA under JSPS-ABC exchange program. He is deeply grateful to IMPA for their hospitality.

References

[1] Hartshorne, R., *Algebraic geometry,* Graduate texts in Mathematics 52, Springer-Verlag, New York, (1977)

[2] Hartshorne, R., *Generalized divisors on Gorenstein curves and a theorem of Noether,* J. Math. Kyoto Univ. 26 (1986), 375–386.

[3] Homma, M. *Singular hyperelliptic curves,* preprint, 1996

[4] Rosenlicht, M., *Equivalence relations on algebraic curves,* Ann. of Math. 56 (1952), 169–191.

[5] Stöhr, K. O., *On the poles of regular differentials of singular curves,* Bol. Soc. Bras. Mat. 24 (1993), 105–136.

Department of Mathematics
Kanagawa University
Rokkakubashi, Kanagawa-ku
Yokohama 221 Japan
E-mail homma@cc.kanagawa-u.ac.jp