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A VARIANT OF A BASE-POINT-FREE PENCIL TRICK AND LINEAR SYSTEMS ON A PLANE CURVE

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Abstract. We prove two variants of a base-point-free pencil trick, which are similar in the spirit of the proof, and apply them to the study of special divisors on a smooth plane curve involving a theorem of Max Noether.

0. Introduction

Max Noether's theorem on the largest possible dimension of a special linear system with a fixed degree on a smooth plane curve is as follows:

Let X be a smooth plane curve of degree $d \geq 4$ over an algebraically closed field, and L an invertible sheaf on X of degree

$$sd - e \quad \textit{with} \quad 1 \leq s \leq d-3 \textit{ and } 0 \leq e < d.$$

Then we have

$$h^0(L) \leq \begin{cases} \frac{1}{2}s(s+1) & \textit{if } s+1 \leq e < d \\ \frac{1}{2}(s+1)(s+2) - e & \textit{if } 0 \leq e \leq s+1 \; . \end{cases}$$

(*Note that two bounds coincide if* $e = s+1$.)

In his paper [2], Hartshorne gave a modern proof of the theorem for any complete irreducible plane curve which may have singular points, and explicitly described the (generalized) linear systems of maximum

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dimension with respect to a fixed degree. His proof of the dimension-estimation part is kind to readers, but that of the additional statements does not seem so kind; honestly speaking, the first author couldn't reach a complete understanding of the proof of the second part. Moreover, his additional statements need a minor modification in the case $e = s+1$, that is, the variety of special divisors $W_{sd-(s+1)}^{\frac{1}{2}s(s+1)-1}(X)$ which parametrizes linear systems of degree $sd-(s+1)$ and projective dimension at least $\frac{1}{2}s(s+1)-1$ has two components if $2 \le s \le d-4$.

In this paper, we will state the correct description of the linear systems of maximal dimension and give a proof of it under assuming Noether's bound, for a smooth plane curve. The idea of our proof is fairly simple; we will use a descending induction by means of the invertible sheaf $O_X(1)$ which corresponds the line sections of X in $\mathbb{P}^2$. In order to work the induction well, we need a variant of a base-point-free pencil trick (Theorem 1.1), which is the topic in Section 1. In Section 2, we will do our task explained above. In Section 3, we will study the $W_{sd-e}^{r(s,e)}(X)$'s, where $r(s,e)$ is the largest possible projective dimension of a linear system of degree $sd-e$. To show the smoothness of $W_{sd-e}^{r(s,e)}(X)$ with $e \neq s+1$, we will use another variant of a base-point-free pencil trick (Lemma 3.1).

1. A variant of a base-point-free pencil trick

The setting of this section is a little more general than that of Introduction. Let X be a smooth curve, which may not have a smooth plane model. An invertible sheaf $O_X(1)$ on X is said to be ample with normal generation if it is ample and natural morphisms

$$H^0(O_X(1))^{\otimes m} \to H^0(O_X(m)) \quad (\, m = 1, 2, \cdots\,)$$

are surjective ([3]). The property is equivalent to the following two conditions:

(i) $O_X(1)$ is very ample; and
(ii) the image of X embedded by the linear system $|O_X(1)|$ is projectively normal.

The invertible sheaf corresponding to the line sections on a smooth plane curve is ample with normal generation.

THEOREM 1.1. *Let $O_X(1)$ be an ample invertible sheaf with normal generation on a complete smooth curve X, and M an invertible sheaf* *on X generated by its global sections. If a two-dimensional subspace V of $H^0(O_X(1))$ generates $O_X(1)$, that is, the linear system $|V|$ is a base-point-free pencil, then the following conditions are equivalent:*

(a) *the natural morphism $H^0(M) \otimes V \to H^0(M(1))$ is surjective;*
(b) $h^1(M(-1)) = 0$.

Proof. To start with, we should explain what a base-point-free pencil trick is.

Let L and M be invertible sheaves on an irreducible smooth curve Y (which may not be complete). Let $s, t \in H^0(Y, L)$ with $(s)_0 \cap (t)_0 = \emptyset$. Here $(s)_0$ and $(t)_0$ mean divisors of the zeros of s and t on Y respectively. Let $u, v \in H^0(Y, M)$ such that $us = vt$ in $H^0(Y, M \otimes L)$. Then there exists $\delta \in H^0(Y, M \otimes L^{-1})$ so that $u = \delta t$ and $v = \delta s$.

We will refer to the argument as the *base-point-free pencil trick.*

First we show the implication (b) $\Rightarrow$ (a). By using the base-point-free pencil trick locally, we have an exact sequence of O_X-modules:

$$0 \to M(-1) \to M \otimes V \overset{(\#)}{\to} M(1) \to 0, \tag{1}$$

where the surjectivity of the morphism (#) comes from the base-point-freeness of $|V|$. From (1), we get an exact sequence

$$H^0(M) \otimes V \to H^0(M(1)) \to H^1(M(-1)).$$

Hence (b) implies (a), which is true without the assumption that the invertible sheaf $O_X(1)$ is ample with normal generation.

Next we show the implication (a) $\Rightarrow$ (b). We start with an extra case; we assume that $M \cong O_X$. Since $H^0(O_X) \otimes V \to H^0(O_X(1))$ is surjective, we have $h^0(O_X(1)) \leq 2$. Hence X is of genus 0 and $O_X(1)$ is of degree 1 because $O_X(1)$ is very ample. Hence we have $h^1(O_X(-1)) = 0$.

Therefore, from now on, we can assume that $M \not\cong O_X$.

The first claim is:

$$H^0(M) \otimes H^0(O_X(m)) \to H^0(M(m))$$

is surjective for each $m \gg 0$. Since M, which is not isomorphic to O_X, is generated by its global sections, we can choose a two-dimensional subspace $W \subseteq H^0(M)$ which generates M, that is $|W|$ is a base-point-free pencil. Hence there is an exact sequence of O_X-modules

$$0 \to O_X(m) \otimes M^{-1} \to W \otimes O_X(m) \to M(m) \to 0.$$

If one chooses an integer m to be large enough, then $H^1(O_X(m)\otimes M^{-1}) = 0$. Hence $W\otimes H^0(O_X(m)) \to H^0(M(m))$ is surjective; and so is $H^0(M)\otimes H^0(O_X(m)) \to H^0(M(m))$.

Now, we choose a basis $\{x_1, x_2; x_3, \ldots, x_n\}$ for $H^0(O_X(1))$ so that $\{x_1, x_2\}$ is a basis for V. Since $O_X(1)$ is ample with normal generation, the vector space $H^0(O_X(\ell))$ is generated by the monomials in $x_1, \ldots, x_n$ of degree ℓ over the base field for any nonnegative integer ℓ. Since $H^0(M) \otimes \langle x_1, x_2\rangle \to H^0(M(1))$ is surjective and $H^0(M)x_j \subseteq H^0(M(1))$ for each j with $1 \le j \le n$, we have

$$H^0(M)x_j \subseteq H^0(M)x_1 + H^0(M)x_2 \tag{2}$$

for each j.

The second claim is:

$$H^0(M(\ell-1)) \otimes V \to H^0(M(\ell))$$

is surjective for $\ell = 1, 2, \ldots$. Let f be a nonzero element of $H^0(M(\ell))$. Let us fix an integer $m \gg 0$. Then we may assume that $H^0(M) \otimes H^0(O_X(\ell+m)) \to H^0(M(\ell+m))$ is surjective by the first claim. Hence the element $fx_1^m \in H^0(M(\ell+m))$ can be represented as

$$fx_1^m = \sum_{\substack{(e_1,\ldots,e_n)\text{with}\\ e_1+\cdots+e_n=\ell+m}} \alpha_{e_1,\ldots,e_n} x_1^{e_1}\cdots x_n^{e_n}, \tag{3}$$

where $\alpha_{e_1,\ldots,e_n}$'s are in $H^0(M)$. By (2), we can rewrite (3) as

$$fx_1^m = \sum_{\substack{(e_1,e_2)\text{with}\\ e_1+e_2=\ell+m}} \beta_{e_1,e_2} x_1^{e_1}x_2^{e_2}.$$

Hence we have

$$x_1^m\left(f - \sum_{j=0}^{\ell} \beta_{m+j,\ell-j}x_1^j x_2^{\ell-j}\right) = \gamma\cdot x_2, \tag{4}$$

where $\gamma \in H^0(M(\ell+m-1))$. Since $(x_1)_0 \cap (x_2)_0 = \emptyset$, the base-point-free pencil trick is applicable to (4). By using that trick successively, we get an element $\delta \in H^0(M(\ell-1))$ so that

$$f - \sum_{j=0}^{\ell} \beta_{m+j,\ell-j}x_1^j x_2^{\ell-j} = \delta x_2,$$

which means f is in the image of the natural map

$$H^0(M(\ell-1)) \otimes V \to H^0(M(\ell)).$$

Hence the second claim has been justified.

By the second claim, we have an exact sequence

$$0 \to H^0(M(\ell-2)) \to H^0(M(\ell-1)) \otimes V \to H^0(M(\ell)) \to 0$$

for each $\ell = 1, 2, \ldots$. Therefore we have

$$h^0(M(\ell)) - h^0(M(\ell-1)) = h^0(M(\ell-1)) - h^0(M(\ell-2))$$

for each ℓ, which is equivalent to the condition

$$h^1(M(\ell-2)) - h^1(M(\ell-1)) = h^1(M(\ell-1)) - h^1(M(\ell)) \tag{5}$$

for each ℓ. Since $h^1(M(\ell)) = 0$ for $\ell \gg 0$, we can conclude that $h^1(M(-1)) = 0$ by (5). □

2. Linear systems of maximal dimension on a plane curve

Throughout this section, we assume that X is a smooth plane curve of degree $d \geq 4$, and denote by $O_X(1)$ the invertible sheaf associated to the line sections of $X \subset \mathbb{P}^2$. We want to describe the invertible sheaves on X which lie on Noether's boundary.

Let L be an invertible sheaf on X with $h^0(L) > 0$, $h^1(L) > 0$ and $L \not\cong O_X$. Then we can write the degree of L as

$\deg L = sd - e$ with $1 \leq s \leq d-3$ and $0 \leq e < d$.

THEOREM 2.1. *Under the above notation, we have:*

(1) *When $e > s+1$, $h^0(L) = \frac{1}{2}s(s+1)$ if and only if there is a positive divisor $Q_1 + \cdots + Q_{d-e}$ of degree $d-e$ on X so that $L \cong O_X(s-1)(Q_1 + \cdots + Q_{d-e})$;*

(2) *When $e = s+1$, $h^0(L) = \frac{1}{2}s(s+1)$ if and only if there is either a positive divisor $Q_1 + \cdots + Q_{d-(s+1)}$ of degree $d-(s+1)$ on X so that $L \cong O_X(s-1)(Q_1 + \cdots + Q_{d-(s+1)})$, or a positive divisor $P_1 + \cdots + P_{s+1}$ of degree $s+1$ on X so that $L \cong O_X(s)(-P_1 - \cdots - P_{s+1})$;*

(3) *When $0 < e < s+1$, $h^0(L) = \frac{1}{2}(s+1)(s+2) - e$ if and only if there is a positive divisor $P_1 + \cdots + P_e$ of degree e on X so that $L \cong O_X(s)(-P_1 - \cdots - P_e)$;*

(4) *When $e = 0$, $h^0(L) = \frac{1}{2}(s+1)(s+2)$ if and only if $L \cong O_X(s)$.*

Proof. First we show the "if" part in each case.

- If $L = O_X(s)$, then $h^0(O_X(s)) = h^0(O_{\mathbb{P}^2}(s))$ because $\deg X > s$; so we have $h^0(O_X(s)) = \binom{s+2}{2}$.
- If $L = O_X(s-1)(Q_1+\cdots+Q_{d-e})$ with $e \geq s+1$, then $h^0(O_X(s-1)(Q_1+\cdots+Q_{d-e})) \geq h^0(O_X(s-1)) = \binom{s+1}{2}$; equality must hold because of Noether's bound.
- If $L = O_X(s)(-P_1-\cdots-P_e)$ with $e \leq s+1$, then $h^0(O_X(s)(-P_1-\cdots-P_e)) \geq h^0(O_X(s)) - e = \binom{s+2}{2} - e$; equality must hold because of Noether's bound.

In order to prove the "only if" parts, we divide the proof into several steps.

Let L be an invertible sheaf for which the equality of Noether's bound holds

Step 1. First we look at the dual $\omega_X \otimes L^{-1}$ of L with respect to the canonical sheaf ω_X. Since X is a plane curve of degree d, the canonical sheaf is $O_X(d-3)$.

If L is in the case (1) of the statement of the theorem, i.e., $\deg L = sd - e$ with $e > s+1$ and $h^0(L) = \frac{1}{2}s(s+1)$, then

$$\deg O_X(d-3) \otimes L^{-1} = (d-2-s)d - (d-e).$$

Note that $0 < d-e < (d-2-s)+1$ because $d > e > s+1$ in our case. Moreover, we have

$$h^0(O_X(d-3) \otimes L^{-1}) = \frac{1}{2}(d-2-s+1)(d-2-s+2) - (d-e)$$

by the Riemann-Roch theorem; therefore $O_X(d-3) \otimes L^{-1}$ is also on Noether's boundary and is in the case (3).

By computing $\deg O_X(d-3) \otimes L^{-1}$ and $h^0(O_X(d-3) \otimes L^{-1})$ in the remaining cases in the same fashion as above, we know that $O_X(d-3) \otimes L^{-1}$ lies on Noether's boundary in each case, more precisely,

- L is in the case (1) $\Leftrightarrow O_X(d-3) \otimes L^{-1}$ is in the case (3)
- L is in the case (2) $\Leftrightarrow O_X(d-3) \otimes L^{-1}$ is in the case (2)
- L is in the case (3) $\Leftrightarrow O_X(d-3) \otimes L^{-1}$ is in the case (1)
- L is in the case (4) $\Leftrightarrow O_X(d-3) \otimes L^{-1}$ is in the case (4).

Step 2. We show that *if* L *is in the cases* (3) *or* (4), *then* $|L|$ *is free from base points.* Indeed, if $|L|$ has a base point P, then $h^0(L(-P)) = h^0(L) = \frac{1}{2}(s+1)(s+2) - e$; but $\deg L(-P) = \deg L - 1 = sd - (e+1)$.

Here $0 < e+1 \leq s+1$, because L is in the cases (3) or (4). These contradict to Noether's bound.

Step 3. In this step, we prove that our assertion is true for the case $s = d-3$ and $0 \leq e \leq d-2$. Since $h^1(L) > 0$, we have $h^0(O_X(d-3) \otimes L^{-1}) > 0$, which means that there is a positive divisor $P_1 + \cdots + P_e$ of degree e on X so that $O_X(d-3) = L(P_1 + \cdots + P_e)$.

Step 4. In this step, we prove the "only if" parts of the cases (3) and (4) by a descending induction on s. In Step 3, we have just treated the case $s = d-3$, which is the first stage of the induction. Since the invertible sheaf L is in the cases (3) or (4), its degree is

$$sd - e \text{ with } 1 \leq s < d-3 \text{ and } 0 \leq e < s+1$$

and

$$h^0(L) = \frac{1}{2}(s+1)(s+2) - e.$$

Let us choose a two-dimensional subspace V of $H^0(O_X(1))$ so that $|V|$ has no base points. By the base-point-free pencil trick, we have an exact sequence

$$(6) \qquad 0 \to H^0(L(-1)) \to H^0(L) \otimes V \overset{(\#)}{\to} H^0(L(1)).$$

Since $|L|$ is free from base points (by Step 2) and $h^1(L(-1)) \geq h^1(L) > 0$, the map $(\#)$ is not surjective by Theorem 1.1. On the other hand, since $\deg L(-1) = (s-1)d - e$ with $0 \leq e \leq (s-1)+1$, we have $h^0(L(-1)) \leq \frac{1}{2}s(s+1) - e$ by Noether's bound. Note that the bound is effective even if $s = 1$, because $e = 0$ or 1 if $s = 1$. Therefore we have

$$(7) \qquad \begin{aligned} h^0(L(1)) &\geq 2(\frac{1}{2}(s+1)(s+2) - e) - (\frac{1}{2}s(s+1) - e) + 1 \\ &= \frac{1}{2}(s+2)(s+3) - e. \end{aligned}$$

Since $\deg L(1) = (s+1)d - e$ with $0 \leq e \leq s$, equality in (7) must hold by Noether's bound. Hence $L(1) \cong O_X(s+1)(-P_1 - \cdots - P_e)$ by the induction hypothesis; and hence $L \cong O_X(s)(-P_1 - \cdots - P_e)$.

Step 5. In this step, we consider the case (1):

$$\deg L = sd - e \quad \text{with} \quad s+1 < e < d$$

and

$$h^0(L) = \frac{1}{2}s(s+1).$$

As saw in Step 1, $O_X(d-3)\otimes L^{-1}$ is in the case (3):

$\deg O_X(d-3)\otimes L^{-1} = (d-2-s)d-(d-e)$ with $0 < d-e < (d-2-s)+1$ and

$$h^0(O_X(d-3)\otimes L^{-1}) = \frac{1}{2}(d-2-s+1)(d-2-s+2)-(d-e).$$

Hence, by Step 4, we have

$$O_X(d-3)\otimes L^{-1} \cong O_X(d-2-s)\left(-\sum_{j=1}^{d-e} Q_j\right)$$

for some points $Q_1, \dots, Q_{d-e} \in X$; and hence $L \cong O_X(s-1)(\sum_{j=1}^{d-e} Q_j)$.

Step 6. Finally we consider the case (2), that is, $\deg L = sd-(s+1)$ and $h^0(L) = \frac{1}{2}s(s+1)$. If $|L|$ has a base point, say Q, then $L(-Q)$ is of degree $sd-(s+2)$ and of dimension $\frac{1}{2}s(s+1)$, which is in the case (1). Hence, by Step 5, we have

$$L(-Q) \cong O_X(s-1)\left(\sum_{j=1}^{d-(s+2)} Q_j\right)$$

for some points $Q_1, \dots, Q_{d-(s+2)} \in X$; and hence we have

$$L \cong O_X(s-1)\left(\sum_{j=1}^{d-(s+1)} Q_j\right),$$

where $Q_{d-(s+1)} = Q$.

Therefore only the remaining case is the case where $\deg L = sd-(s+1)$, $h^0(L) = \frac{1}{2}s(s+1)$ and $|L|$ has no base points. When $s = d-3$, it is done in Step 3, and when $s = 1$, it can be reduced to the case $s = d-3$ by considering the dual series like Step 5. So we may assume that $1 < s < d-3$. Since $\deg L(-1) = (s-1)d-(s+1)$, the largest possible dimension of $H^0(L(-1))$ is $\frac{1}{2}(s-1)s$. If $h^0(L(-1))$ takes this value, then $L(-1) \cong O_X(s-2)(\sum_{j=1}^{d-(s+1)} Q_j)$ by Step 5. Hence we may assume that

$$h^0(L(-1)) \le \frac{1}{2}(s-1)s-1.$$

On the other hand, since $|L|$ has no base points, we can consider the exact sequence (6) for our L and the map (#) is not surjective. Hence we have

$$\begin{aligned} h^0(L(1)) &\geq 2\left(\frac{1}{2}s(s+1)\right) - \left(\frac{1}{2}(s-1)s - 1\right) + 1 \\ &= \frac{1}{2}(s+2)(s+3) - (s+1). \end{aligned}$$

Since $\deg L(1) = (s+1)d - (s+1)$, we have

$$L(1) \cong O_X(s+1)\left(-\sum_{j=1}^{s+1} P_j\right)$$

by Step 4. □

3. The varieties of special divisors

We keep the same notation and assumption as in the previous section. Furthermore, we use the following notation:

$$r(s,e) := \begin{cases} \frac{1}{2}s(s+1) - 1 & \text{if } s+1 \leq e < d \\ \frac{1}{2}(s+1)(s+2) - e - 1 & \text{if } 0 \leq e \leq s+1\ . \end{cases}$$

The aim of this section is to study the scheme $W^{r(s,e)}_{sd-e}(X)$ of special divisors of degree $sd - e$ and dimension at least $r(s,e)$; however, the dimension of any invertible sheaf which corresponds to a closed point of the scheme is exactly $r(s,e)$ because of Noether's bound.

To study the $W^{r(s,e)}_{sd-e}(X)$'s with $e \neq s+1$, we need a lemma, which is similar to Theorem 1.1 in the spirit of the proof.

LEMMA 3.1. *Let s be an integer with $1 \leq s \leq d-4$, and D an effective divisor on X of degree $e \leq s$, or of degree $s+1$ with $s \geq 2$ such that $|O_X(s)(-D)|$ is base-point-free. Then the natural map*

$$H^0(O_X(s)(-D)) \otimes H^0(O_X(1)) \to H^0(O_X(s+1)(-D))$$

is surjective.

Proof. By Noether's bound, the linear system $|O_X(s)(-D)|$ is base-point-free even if D is in the former case, and $O_X(s)(-D) \not\cong O_X$ because

$\deg O_X(s)(-D) = sd - e > 0$. Hence we can apply the first claim in the proof of Theorem 1.1 to $O_X(s)(-D)$ and we know that

(8) $$H^0(O_X(s)(-D)) \otimes H^0(O_X(m)) \to H^0(O_X(s+m)(-D))$$

is surjective for $m \gg 0$.

Choose a basis $\{x_1, x_2, x_3\}$ of $H^0(O_X(1))$ so that $(x_1)_0 \cap (x_2)_0 = \emptyset$, and let V be its subspace generated by x_1 and x_2. Let us consider the commutative diagram:

$$\begin{array}{ccc} H^0(O_X(s)(-D)) \otimes H^0(O_X(1)) & \bar{\alpha} & \\ & \searrow & \\ \uparrow & & H^0(O_X(s+1)(-D)) \\ & \nearrow & \\ H^0(O_X(s)(-D)) \otimes V & \alpha & \end{array}$$

First suppose that $\mathrm{Im}\alpha = \mathrm{Im}\bar{\alpha}$, which means

$$H^0(O_X(s)(-D))x_3 \subseteq H^0(O_X(s)(-D))x_1 + H^0(O_X(s)(-D))x_2.$$

Hence, taking account of (8), we can conclude that the linear map α (and also $\bar{\alpha}$) is surjective by the same argument to the second claim of Theorem 1.1. (But, actually, it is impossible in our case because of Theorem 1.1 itself.)

Therefore, we may assume that

(9) $$\dim \mathrm{Im}\bar{\alpha} \geq \dim \mathrm{Im}\alpha + 1.$$

Thanks to the base-point-free pencil trick, we have an exact sequence

$$\begin{aligned} 0 \quad \to \quad & H^0(O_X(s-1)(-D)) \\ \to \quad & H^0(O_X(s)(-D)) \otimes V \xrightarrow{\alpha} H^0(O_X(s+1)(-D)). \end{aligned}$$

On the other hand, by Noether's bound (or Theorem 2.1), we have

$$\begin{aligned} h^0(O_X(s)(-D)) &= \frac{1}{2}(s+1)(s+2) - e \\ h^0(O_X(s+1)(-D)) &= \frac{1}{2}(s+2)(s+3) - e \end{aligned}$$

because $0 \leq e \leq s$. Moreover we can show that

$$h^0(O_X(s-1)(-D)) = \frac{1}{2}s(s+1) - e;$$

it is obvious if $0 \leq e \leq s$ because of Noether's bound. When $e = s+1$, a possible values of $h^0(O_X(s-1)(-D))$ is $\frac{1}{2}s(s+1) - (s+1) = \frac{1}{2}(s-1)s - 1$ or $\frac{1}{2}(s-1)s$. If the latter case occur, then $O_X(s-1)(-D) \cong$

$O_X(s-2)(E)$ for some effective divisor E of degree $d-(s+1)$; hence $O_X(s)(-D) \cong O_X(s-1)(E)$. Hence $|O_X(s)(-D)|$ has a base point because of Noether's bound, which contradicts the assumption on D.

Therefore we have

$$h^0(O_X(s+1)(-D)) = \dim \operatorname{Im}\alpha + 1.$$

Hence $\bar{\alpha}$ is surjective by (9). $\square$

THEOREM 3.2. *Let s be an integer with $1 \le s \le d-3$. Then the scheme of special divisors $W^{r(s,e)}_{sd-e}(X)$ with $e \ne s+1$ is a smooth variety of dimension e if $0 \le e < s+1$, or of dimension $d-e$ if $s+1 < e < d$, respectively.*

Proof. Since $L \mapsto O_X(d-3) \otimes L^{-1}$ gives an isomorphism between $W^{r(s,e)}_{sd-e}(X)$ and $W^{r(d-2-s,d-e)}_{(d-2-s)d-(d-e)}(X)$ if $e \ne 0$ (see, Step 1 of the proof of Theorem 2.1), we may assume that $0 \le e < s+1$.

Let $S^e(X)$ be the e-fold symmetric product of the curve X. Then the morphism

$$\varphi_- : S^e(X) \to W^{r(s,e)}_{sd-e}(X)$$

defined by $\varphi_-(D) = O_X(s)(-D)$ is surjective by Theorem 2.1. In particular, $W^{r(s,e)}_{sd-e}(X)$ is irreducible. Moreover the morphism φ_- is injective. Indeed, $\varphi_-(D) = \varphi_-(D')$ implies the isomorphism $O_X(D) \cong O_X(D')$, which means that $D = D'$ because $\deg D = e < s+1 \le d-2$. Hence $\dim W^{r(s,e)}_{sd-e}(X) = e$.

Since $W^{r(s,e)+1}_{sd-e}(X) = \emptyset$ by Noether's bound, the Zariski tangent space

$$T_{\varphi_-(D)}(W^{r(s,e)}_{sd-e}(X))$$

to the scheme $W^{r(s,e)}_{sd-e}(X)$ at $\varphi_-(D)$ is isomorphic to $(\operatorname{Im}\mu_0)^{\perp}$, where

$$\mu_0 : H^0(O_X(s)(-D)) \otimes H^0(O_X(d-3-s)(D)) \to H^0(O_X(d-3)) = H^0(\omega_X).$$

As for those matters, see [1, IV, (4.2)].

Let us consider the commutative diagram

$$\begin{array}{ccc} H^0(O_X(s)(-D)) \otimes H^0(O_X(d-3-s)) & \xrightarrow{\mu_1} & H^0(O_X(d-3)(-D)) \\ \downarrow & & \downarrow \iota \\ H^0(O_X(s)(-D)) \otimes H^0(O_X(d-3-s)(D)) & \xrightarrow{\mu_0} & H^0(O_X(d-3)) \end{array}$$

where two vertical maps in the diagram come from the natural injective morphism of sheaves $O_X(-D) \hookrightarrow O_X$. Since

$$H^0(O_X(s)(-D)) \otimes H^0(O_X(1))^{\otimes(d-3-s)} \to H^0(O_X(d-3)(-D))$$

is surjective by Lemma 3.1, so is μ_1. On the other hand, $h^0(O_X(d-3)(-D)) = g - e$ by Noether's bound, where $g = \frac{1}{2}(d-1)(d-2)$ is the genus of the curve X. Therefore, since ι is injective, we have

$$\dim \operatorname{Im} \mu_0 \geq g - e,$$

which means

$$\dim T_{\varphi_-(D)}(W_{sd-e}^{r(s,e)(X)}) = \dim(\operatorname{Im} \mu_0)^{\perp} \leq e.$$

But we already know that $\dim W_{sd-e}^{r(s,e)}(X) = e$. Hence the irreducible scheme is smooth at $\varphi_-(D)$. □

For the case $e = s + 1$, we have

Proposition 3.3. *The underlying topological space $W_{d-2}^0(X)$ (which is the case $s = 1$) and that of $W_{(d-3)d-(d-2)}^{\frac{1}{2}(d-3)(d-2)-1}(X)$ (which is the case $s = d-3$) are irreducible of dimension $d-2$, but for $2 \leq s \leq d-4$, that of $W_{sd-(s+1)}^{\frac{1}{2}s(s+1)-1}(X)$ has exactly two components; one is of dimension $s+1$, the other is of dimension $d-(s+1)$ and the intersection of these two components is of dimension 2.*

Proof. Let us consider two natural morphisms

$$\begin{aligned} \varphi_+ : S^{d-(s+1)}(X) &\to W_{sd-(s+1)}^{\frac{1}{2}s(s+1)-1}(X) \\ D &\mapsto O_X(s)(D) \end{aligned}$$

and

$$\begin{aligned} \varphi_- : S^{s+1}(X) &\to W_{sd-(s+1)}^{\frac{1}{2}s(s+1)-1}(X) \\ D &\mapsto O_X(s)(-D) \end{aligned}$$

which are injective. By Theorem 2.1 (2), $W_{sd-(s+1)}^{\frac{1}{2}s(s+1)-1}(X)$ is covered with the images of φ_+ and φ_-, which are irreducible of dimension $d-(s+1)$ and $s+1$, respectively. If $L \in \operatorname{Im}\varphi_+ \cap \operatorname{Im}\varphi_-$, then we can choose points

$$P_1, \ldots, P_{s+1}; Q_1, \ldots, Q_{d-(s+1)} \in X$$

so that

$$L \cong O_X(s)\left(-\sum_{i=1}^{s+1} P_i\right) \cong O_X(s-1)\left(\sum_{j=1}^{d-(s+1)} Q_j\right).$$

Hence the divisor $\sum_{i=1}^{s+1} P_i + \sum_{j=1}^{d-(s+1)} Q_j$ must be a line section of $X \subset \mathbb{P}^2$. Conversely, for each line section $\sum_{i=1}^{s+1} P_i + \sum_{j=1}^{d-(s+1)} Q_j$,

$$O_X(s)(-\sum_{i=1}^{s+1} P_i) \cong O_X(s-1)\left(\sum_{j=1}^{d-(s+1)} Q_j\right)$$

and

$$h^0(O_X(s)\left(-\sum_{i=1}^{s+1} P_i)\right) = h^0(O_X(s)) - (s+1)$$

by Noether's bound, which means those two invertible sheaves correspond to a point of $\mathrm{Im}\varphi_+ \cap \mathrm{Im}\varphi_-$. Therefore, we have that

$$\mathrm{Im}\varphi_- \subset \mathrm{Im}\varphi_+ \quad \text{if } s = 1;$$

$$\mathrm{Im}\varphi_+ \subset \mathrm{Im}\varphi_- \quad \text{if } s = d-3;$$

$$\mathrm{Im}\varphi_- \not\subset \mathrm{Im}\varphi_+ \text{ and } \mathrm{Im}\varphi_+ \not\subset \mathrm{Im}\varphi_- \quad \text{if } 1 < s < d-3.$$

REMARK.

(1) It is easy to show that $W^0_{d-2}(X)$ and $W^{\frac{1}{2}(d-3)(d-2)-1}_{(d-3)d-(d-2)}(X)$ are smooth varieties, which are isomorphic to each other.

(2) By using Lemma 3.1, we can show that $W^{\frac{1}{2}s(s+1)-1}_{sd-(s+1)}(X)$ with $2 \leq s \leq d-4$ is smooth at each point of the outside of $\mathrm{Im}\varphi_+ \cap \mathrm{Im}\varphi_-$, like Theorem 3.2.

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