

## A simple approach for the fundamental period of MDOF structures

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### 1. Introduction

The fundamental period is a key parameter for the seismic design of a building structure using the equivalent-lateral-force procedure; in principle, it can be accurately evaluated by means of an eigenvalue analysis [1] on a structural model. In most building-design projects, since the building's period cannot be analytically calculated before it has been designed, accurate computation is generally not possible in the preliminary design stage, and, typically, simple formulae for the fundamental period are used to initiate the design process. These simple formulae also serve as a basis for limiting the period from a finite-element model by applying the upper-bound factor suggested in the 2003 NEHRP Recommended Provisions for Seismic Regulations for New Buildings and subsequently in ASCE 7-05 [2]. Therefore, at present, simple formulae for estimating the fundamental period with good accuracy play an important role in structural design [3, 4].

Many researchers have previously proposed such formulae for this purpose. Generally, there are two kinds of simple formulae for the fundamental period: empirical [3-16] and analytical [17-21]. A lot of empirical formulae have been developed. Asteris et al. [5, 6] give an extensive review of these formulae. Empirical formulae adopted in most codes are simply expressed in terms of the height of buildings [19, 21]. Some researchers take into account other parameters apart from the height of building. Kose [9] takes into account the presence of infill walls and frame type. Hatzigeorgiou and Kanapitsas [11] proposed an expression considering the soil flexibility, the influence of shear walls, and the external and internal infill wall. Asteris et al. [5, 6] proposed a more accurate formula that takes into account the number of stories, the number of span, the span length, the infill wall panel stiffness and the percentage of openings within the infill wall. Further, Asteris et al [8] recognized that the vertical geometric irregularity significantly influences the fundamental period, and proposed a reduction factor to quantify this effect.

Analytical formulae also have been adopted in many codes [19-21]. This study focuses on the analytical ones, which have generally been developed based on vibration theory for a multiple-degree-of-freedom (MDOF) system. Among these, Rayleigh's method, Geiger's method, and Dunkerley's method are the three most widely used; the first two of which were specified in the 1997 Uniform Building Code [19], the Japanese seismic code [20], respectively. In this paper, a new, simpler, and more

accurate method for estimating the fundamental period of a MDOF system is proposed.

### 2. The proposed method for estimating the fundamental period

A simple method for estimating the fundamental period of an MDOF system is proposed in this chapter. The basic principle is to replace a complicated MDOF system with an equivalent SDOF system for which the fundamental period can be easily obtained. To realize the SDOF-system equivalence, a procedure to replace a two-degree-of-freedom (2-DOF) system with an SDOF system having the same fundamental period, called the two-to-single (TTS) procedure, is developed firstly; then, using the TTS procedure successively, the MDOF system can be replaced with an equivalent SDOF system having approximately the same fundamental period.

#### 2.1 A procedure to replace a 2-DOF system with an SDOF system

In order to develop the TTS procedure to reduce a 2-DOF system to an SDOF system with the same fundamental period, a 2-DOF system and an equivalent SDOF system are considered, as shown in Fig. 1. In essence, developing the TTS procedure means expressing parameters including mass,  $m_{eq}$ , and stiffness,  $k_{eq}$ , of the equivalent SDOF system in terms of the parameters of the 2-DOF system. For this purpose, the following two equivalent equations are considered

$$m_{eq} = m_1 + m_2 \quad (1)$$

$$T_{eq} = T_{2-DOF} \quad (2)$$

here,  $m_i$ ,  $i = 1, 2$ , is mass of the  $i$ th degree of freedom and  $T_{2-DOF}$  is the fundamental period of the 2-DOF system;  $T_{eq}$  is the fundamental period of the equivalent SDOF system. In order to determine the stiffness,  $k_{eq}$ , of the equivalent SDOF system using Eq. (2), the fundamental period,  $T_{2-DOF}$ , of the 2-DOF system should be derived firstly.

Consider the 2-DOF system in free harmonic vibration. The basic eigen problem for this system is represented as

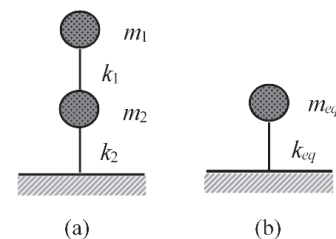


Fig. 1. Illustration of the concept of replacing a 2-DOF system with an equivalent SDOF system

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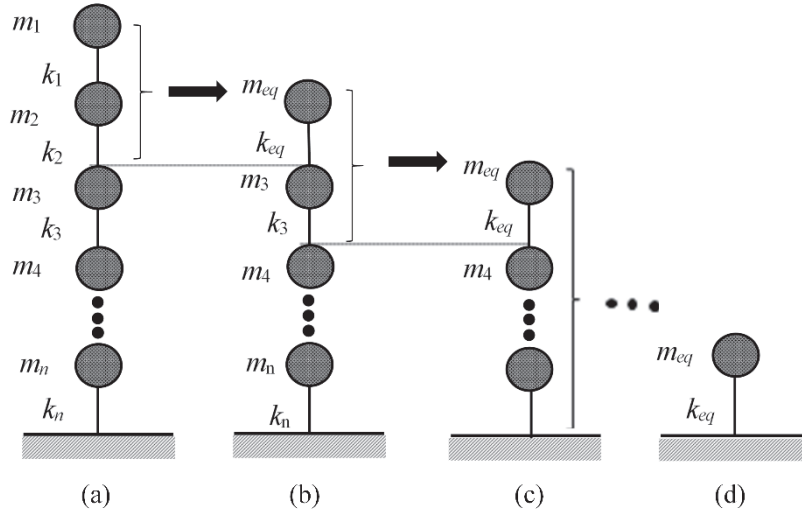


Fig. 2. Illustration of the concept of replacing an MDOF system with an equivalent SDOF system

$$(\omega^2 [M] - [K])\{u\} = 0 \quad (3)$$

where  $\omega_i$ ,  $i = 1, 2$ , are the free-vibration frequencies,  $[M]$  and  $[K]$  are the mass and stiffness matrices of the 2-DOF system, respectively, and are expressed as

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad [K] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}$$

and  $k_i$ ,  $i = 1, 2$ , is the stiffness of the  $i$ th degree of freedom.

By eigenvalue analysis, the fundamental frequency  $\omega_1$  can be given by

$$\omega_1^2 = \frac{1}{2} \left[ \frac{k_1 + k_2}{m_2} + \frac{k_1}{m_1} - \sqrt{\left( \frac{k_1 + k_2}{m_2} - \frac{k_1}{m_1} \right)^2 + 4 \frac{k_1^2}{m_1 m_2}} \right] \quad (4)$$

As  $T_{2-DOF} = 2\pi/\omega_1$ , according to Eq. (2), the stiffness,  $k_{eq}$ , of the SDOF system is given by

$$k_{eq} = \frac{(m_1 + m_2)}{2} \left[ \frac{k_1 + k_2}{m_2} + \frac{k_1}{m_1} - \sqrt{\left( \frac{k_1 + k_2}{m_2} - \frac{k_1}{m_1} \right)^2 + 4 \frac{k_1^2}{m_1 m_2}} \right] \quad (5)$$

Using Eq. (1) and (5), an equivalent SDOF system having the same fundamental period as the 2-DOF system can be obtained.

## 2.2 A procedure for estimating the fundamental period of an MDOF system

Successively using the procedure for replacing a 2-DOF system with an equivalent SDOF system as described above, a procedure for finding the fundamental period of an MDOF system can be developed. The concept of this procedure is illustrated in Fig. 2. And, the procedure includes following steps:

1. For the MDOF system shown in Fig. 2(a), the top two masses  $m_1$  and  $m_2$  are assumed to lie on rigid ground and can be considered as a 2-DOF system. Then, based on the TTS procedure (i.e., Eq. (1) and (5)), an equivalent SDOF system having the same fundamental period as the top 2-DOF system can be obtained, forming a new MDOF system as shown in Fig. 2(b).

2. Then, as in step (1), the top two masses of the new MDOF system as shown in Fig. 2(b) are considered as a new 2-DOF system lying on rigid ground and can be replaced with another equivalent SDOF system using Eq. (1) and (5) again, forming another new MDOF system, as shown in Fig. 2(c).

3. By application of the TTS procedure successively to the remaining lower masses, finally, the MDOF system is replaced with an equivalent SDOF system, as shown in Fig. 2(d). Then, the fundamental period can be readily obtained.

## 2.3 Validation of the rigid-ground assumption

In the procedure for replacing an MDOF system with an equivalent SDOF system described in the previous section, at each step of replacement, the top two masses are always considered as a 2-DOF system lying on rigid ground. However, except at the final step, the 2-DOF system lies on a floor with limited stiffness. In order to validate the rigid-ground assumption, the fundamental periods of a large number of MDOF structures are computed using the procedure described in Section 2.2 and compared with those obtained using an eigenvalue analysis.

The analyzed MDOF structures are divided into two major categories: MDOF structures with floor stiffness varying with height and those with only one special floor with different stiffness from the others. As the mass of the actual structure generally varies less significantly as a function of height than does stiffness, the mass,  $m_0$ , of the analyzed structures is considered constant.

In the first category, the variation of stiffness with height is expressed as

$$k_i = r^{i-1} k_0 \quad (6)$$

where  $k_i$  is the stiffness of the  $i$ th mass point, as shown in Fig. 2(a),  $k_0$  is a constant value, and factor  $r$  represents the variation degree of stiffness along height. Eq. (6) means that, the stiffness of the top story equals  $k_0$ , and stiffness of any lower  $i$ th story is  $r$  times as large as that of the upper

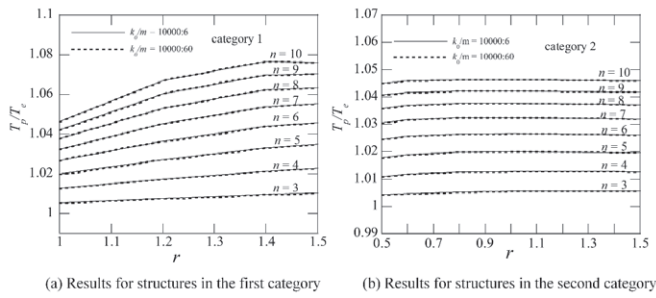


Fig. 3. Comparison of the fundamental periods obtained by the procedure described in Section 2.2 with those obtained by eigenvalue analysis

$i$ -th story. Generally, as the stiffness of the actual structure increases from the top to the bottom, factor  $r$  is considered to vary from 1 to 1.5.

In the second category, the stiffness of only a special floor,  $k_i$ , is considered variable, and the others are constant and equal to  $k_0$ . The variation of the stiffness of this special floor is expressed as

$$k_i = rk_0 \quad (7)$$

Eq. (7) means that, stiffness of the special story is  $r$  times as large as that of other stories equaling  $k_0$ . In this case, factor  $r$  is considered to vary from 0.5 to 1.5, and  $i$  varies from 1 to  $n$ , where  $n$  the number of stories.

It can be easily shown that, in these designed MDOF structures, the parameters controlling the fundamental period are the factor  $r$ , the ratio between stiffness and mass,  $k_0/m_0$ , and the number of stories  $n$ . Thus, the error in the estimated fundamental period caused by the rigid-ground assumption is also considered to be affected by these three parameters. The variation ranges of the parameter  $r$  have been introduced above, for the parameter  $k_0/m_0$ , two values, 10,000 (kN/cm)/6 (t) and 10,000 (kN/cm)/60 (t), are considered in the following calculation. The value, 10,000 (kN/cm)/6 (t), is determined according to an actual structure constructed in Japan [22]. To observe the possible effect of the parameter  $k_0/m_0$  on the error clearly, another extreme value, 10,000 (kN/cm)/60 (t), is assumed. The extreme range assumed for the parameter  $k_0/m_0$  is to observe the possible effect clearly instead of representing actual condition. And number of stories  $n$  is considered to vary from 3 to 10.

The fundamental periods of these MDOF structures are calculated using the procedure described in Section 2.2 and compared against those obtained using an eigenvalue analysis. The errors are expressed by ratios of the fundamental periods calculated by the procedure in Section 2.2,  $T_p$ , with those by an eigenvalue analysis,  $T_e$ . Fig. 3(a) shows the results of the first category of MDOF structures. For the second category of MDOF structures, results are very similar regardless of the value of  $i$  expressed in Eq. (7); for simplicity, only representative results when  $i = n$  are shown in Fig. 3(b). In these figures, the horizontal coordinate is the factor  $r$ , representing the variation degree of stiffness, and the longitudinal coordinate represents the error.

It is observed that, for both subcategories in which there is error in the estimated fundamental period, the maximum relative error is less than 8%. The errors are dependent on the factor  $r$  and the number of stories  $n$ , but

not on the ratio  $k_0/m_0$ . The errors increase with increasing  $r$  for the first category but do not change noticeably for the second category. For both subcategories, the errors increase with  $n$ . Comparing the effects of  $n$  and  $r$  on the errors, that of  $n$  is clearly more prominent.

The reason for the dependence on the number of stories is that, when replacing an MDOF system with an equivalent SDOF system, the top 2-DOF system at each step is assumed to lie on rigid ground, when in fact it lies on a floor with limited stiffness; thus, the more stories the analyzed MDOF system has, the more the assumptions used, resulting in a larger error.

Generally speaking, the rigid-bedrock assumption used in the procedure described in Section 2.2 can cause a calculation error in the fundamental period, but the maximum relative error of the analyzed MDOF structures is below about 8%. The errors are affected by the number of stories  $n$  and the variation degree of the stiffness with height, although the former effect is more significant.

#### 2.4 Correction factor

Based on the analysis in the previous subsection, the prediction of the fundamental period using the procedure described in Section 2.2 is improved with the appropriate introduction of a correction factor.

The fact that the error in the fundamental period obtained using the procedure in Section 2.2 is affected by the number of stories and the variation degree of the stiffness along height leads us to conclude that the correction factor should be expressed in terms of the number of stories,  $n$ , and a factor representing the variation degree of stiffness. However, since the variation degree of the stiffness of an actual building cannot be expressed as a single factor like the idealized one,  $r$ , used previously, and since an increase in the number of stories affects the error more significantly than variation of the stiffness, the correction factor is expressed only in terms of  $n$ .

To isolate the effects of variations of stiffness and mass, MDOF structures with constant mass and stiffness with height are used to conduct the correction. MDOF structures composed of 3–20 stories are used for the correction. Then, a correction factor  $R$  is introduced, defined as the ratio between the fundamental periods obtained by an eigenvalue analysis and by the procedure described in Section 2.2. To determine the correction factor  $R$ , the fundamental ratios of the exact and predicted periods of all analyzed MDOF structures are computed, and the results are shown in

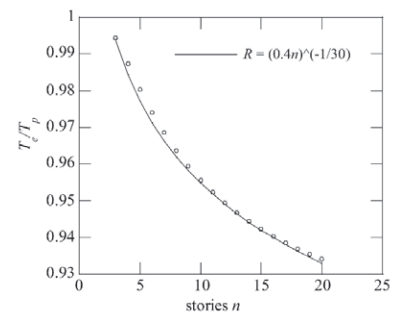


Fig. 4. Ratios between the fundamental periods obtained by eigenvalue analysis and by the method described in Section 2.2

Fig. 4. By trial-and-error analysis of a large number of functional forms, a very simple function is adopted for the correction factor  $R$ , given by

$$R = (0.4n)^{-1/30} \quad (8)$$

The accuracy of this function can also be found very well from Fig. 4. The standard deviation of residuals expressing the random variability of results by Eq. (8) is almost equal to 0.001.

Finally, considering the correction factor, the fundamental period of an MDOF structure can be estimated as

$$T_{pr} = 2\pi R \sqrt{\frac{m_{eq}}{k_{eq}}} \quad (9)$$

where  $m_{eq}$  and  $k_{eq}$  are the mass and stiffness, respectively, of the final equivalent SDOF system obtained by the procedure in Section 3.2.

The proposed method is composed of three equations (i.e., Eq. (1), (5), and (8)), of which the second equation seems more complicated than the current methods at first glance. In Rayleigh's method, the mode shape should be determined first; and, in Geiger's method, the top displacement should be estimated. As Eq. (5) is expressed in terms of only mass and stiffness without any other additional parameters, the proposed method is considered simpler and more direct than any current methods.

It should be noted that, the proposed method is developed for estimation of the fundamental period of the widely used MDOF structural model. This means that, for an actual structure, it must be simplified as an MDOF model before applying the proposed method. During the simplification, besides the structural elements, the infill walls also should be properly considered in the model, since contribution of the infill walls to the fundamental period may be also crucial [5, 6].

In addition, as the proposed method considers variations of mass and stiffness with height, thus the method is available for structures with vertical irregularity. For structures with plan irregularities, torsion may be caused to the building, thus torsional stiffness should be considered in the model of the structure. However, during the derivation of the proposed method, only lateral stiffness is considered. Thus, the proposed method is only available for the shear-type MDOF system. Improving the proposed method to analyze structures with plan irregularities is necessary in the further study.

#### 4. Examples using the proposed method

In order to investigate the accuracy of the proposed method, a recalculation of the fundamental periods of the two categories of MDOF structures introduced in Section 2.3 is performed, and fundamental period ratios between the predicted periods,  $T_{pr}$ , and the exact ones are shown in Fig. 5. It is observed that errors are very low for both categories, with the maximum relative error below 3%. Although the error increases with the number of stories for the first category of MDOF structures, the error level (3%) is considered acceptable for engineering use.

In addition, in order to compare the accuracy of the proposed method with current methods, the fundamental periods of the two categories of MDOF structures are also estimated by the current methods. MDOF

structures with as many as 60 stories are considered for comparison. Representative results are shown in Figs. 6 (a)-(d). In these figures, the horizontal coordinate is  $n$  and the longitudinal coordinates are the fundamental periods calculated by different methods.

It can be noted that all results obtained by the proposed method are much more accurate than those obtained by Dunkerley's method, the Eurocode 8 method and Geiger's method adopted in Japanese code. Indeed, the accuracy of the proposed method is nearly equivalent to that of Rayleigh's method adopted in UBC 1997.

Generally speaking, the accuracy of the proposed method is very good and is much better than that of Dunkerley's method and the Eurocode 8 method. For most of the estimated structures, the accuracy of the proposed method is better than those of Rayleigh's method adopted in UBC 1997 and Geiger's method adopted in Japanese code.

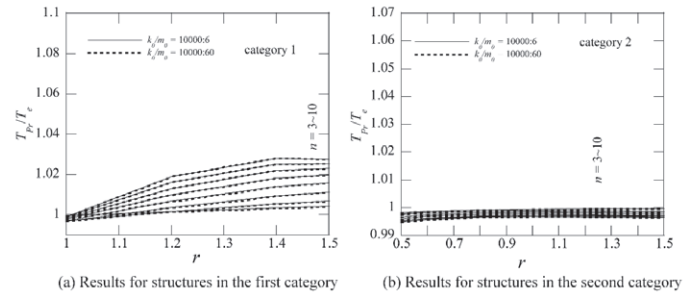


Fig. 5. Comparison between fundamental periods obtained by the proposed method and by eigenvalue analysis

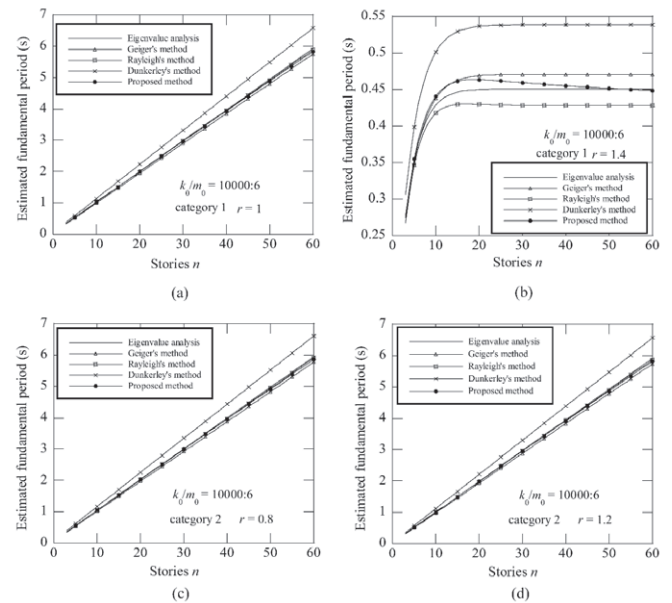


Fig.6. Fundamental periods calculated by different methods. (a) Results for structures in the first category when  $r = 1$ . (b) Results for structures in the first category when  $r = 1.4$ . (c) Result for structures in the second category when  $r = 0.8$ . (d) Results for structures in the second category when  $r = 1.2$

#### 5. Conclusions

On the basis of the preceding discussion, one can draw the following conclusions:

A simple method of evaluating the fundamental period by replacing the complicated MDOF system with an equivalent SDOF system is proposed. The proposed method is available for shear-type MDOF system. As the proposed method is composed of three simple explicit formulae, it can be conveniently implemented in simple spreadsheets. In addition, the application of the proposed method does not require expert knowledge concerning eigenvalue analysis; thus, the proposed method is thought can be used by practicing engineers conveniently. Moreover, as simple formulae are expressed in terms of the mass, stiffness, and number of stories directly without the mode shape or top displacement, the proposed method is a simpler and a more direct method.

The accuracy of the proposed method is investigated by estimating a series of designed MDOF structures and 19 MDOF models of actual structures, and is found to be reasonably good. The accuracy of the proposed method is much better than that of Dunkerley's method and the Eurocode 8 method, and is better than that of Rayleigh's method adopted in UBC 1997 and Geiger's method adopted in Japanese code for most of the analyzed structures.

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