# The Independent Goods and The Aggregation of Commodities

by

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**Abstract**

In this paper, the generalized concept of Sono's independent goods is presented. This paper makes it clear that this concept is mathematically equivalent to the strong separability in demand analysis and discusses its application for the aggregation problem. Main conclusion is that, if we would discuss some economic problem by means of an additive separable utility function, it is enough for us to employ the independent goods with respect to a partition of goods.

**Introduction**

This paper deals with the relationship between the concept of separability and the independent goods, of which the latter was made research by Sono(8). As is well-known, Hicks(4) had an argument about a theory of groups of commodities, in chapter 2 of his principal book, to get an answer for aggregation problem. Sono(8) and Leontief(5) showed in their respective ways some conditions for the possibility of demand aggregation such that the aggregation function or the market demand function must satisfy, introducing a notion of the independent goods and the concept of separability.

One side, a more sophisticated definition of separability in term of utility function was given by Goldman and Uzawa(2). They formulated it in term of an individual utility function and clarified its relation to the form of the utility function.

In this paper, we shall define a generalized concept of independent goods and consider its connection with the separability and the aggregation of commodities. The plan of this paper is as follows. In section 1, we shall consider the relationship between the separability and the Sono's independent goods in term of utility function. In there, it will be shown that the Sono's independent goods implies the "weak" separability in the sense of Goldman and Uzawa(2). Section 2 deals with the generalized concept of the independent goods and its equivalent property to the "strong" separability. In section 3, some discussions on the generalized independent goods in the aggregation problem are presented. Section 4 is a concluding remark.

1. WE ARE CONCERNED with a finite number of commodities which will be labelled $i = 1, 2, 3, \ldots, n$. The set of all $n$ commodities will be denoted by $N$. Let $(N_r, N_s, N_t)$ be a partition of the set $N$ and they hold the following conditions such that

$$N_r \cup N_s \cup N_t = N,\ N_\lambda \cap N_\kappa \text{ is empty for all } \lambda \neq \kappa\ (\lambda, \kappa = r, s, t)$$

We assume that a utility function of some individual $U(x)$ defined on the commodity space is continuously twice-differentiable and has a positive marginal utility everywhere. We further assume that each indifference surface may be connected by a differentiable path and convex toward the origin.

*Definition 1 (Strong separability)*: The utility function $U(x)$ is called "strongly separable" with respect to a partition $(N_r, N_s, N_t)$ if $U(x)$ fulfilles the following conditions such that

$$\frac{\partial}{\partial x_k}\left(\frac{U_i}{U_j}\right) = 0, \text{ for all } i \in N_r, j \in N_s, k \in N_t \qquad \ldots.(1)$$

where $U_i$ stands for the partial derivative of $U(x)$ with respect to $x_i$, and $x_i$ is the quantity of $i$-th good as same follows.

*Definition 2 (Weak separability)*: The utility function $U(x)$ is called "weak separable" with respect to a partition $(N_r, N_s, N_t)$ if $U(x)$ fulfilled the following conditions such that

$$\frac{\partial}{\partial x_k}\left(\frac{U_i}{U_j}\right) = 0, \text{ for all } i, j \in N_\lambda, k \notin N_\lambda, i \neq j, \lambda = (r, s, t) \quad \ldots.(2)$$

The above definition have a same sort of characteristic of those which were given by Goldman and Uzawa(2). Obviously, the strong separability defined in the formula(1) is a peculiar case defined in the formula(2). It is noted that the definition of the strong separability needs than three goods, because the each subset $N_r$, $N_s$, $N_t$ of $N$ must be non-empty.

Now, it is in our knowledge that the following assertions are concluded that

1. If the utility function $U(x)$ satisfied the conditions given by def. 1, then $U(x)$ is of the form such that

$$U(x) = F(U^r(x^r) + U^s(x^s) + U^t(x^t)) \quad \ldots\ldots\ldots (3)$$

2. If the utility function $U(x)$ satisfied the conditions given by def. 2, then $U(x)$ is of the form such that

$$U(x) = G(U^r(x^r), U^s(x^s), U^t(x^t)) \quad \ldots\ldots\ldots (4)$$

provided that $x^r$, $x^s$, $x^t$ are sub-vectors of $x$ such that $x^r = (x_i)$, $i \in N_r$ and so on. And $F(\cdot)$ is a monotone increasing function of one variable and $G(\cdot)$ is of three variables. (For investigating closely, see Goldman and Uzawa(2).)

*Definition 3*: Let $N_r = (1, 2)$, $N_s \cup N_t = (3, 4, \ldots, n)$ be a partition of the set $N$. Good 1 and good 2 are also called "independent of $N_s \cup N_t$ if the utility function $U(x)$ fulfilled the conditions with respect to a partition $(N_r, N_s, N_t)$ such that

$$\frac{\frac{\partial}{\partial x_l}\left(\frac{U_i}{U_k}\right)}{\frac{U_i}{U_k}} = \frac{\frac{\partial}{\partial x_l}\left(\frac{U_j}{U_k}\right)}{\frac{U_j}{U_k}} = \theta_l^k(x_1, x_2), \text{ for all } l, k \in N_r, i, j \in N_s \cup N_t \quad (5)$$

The formula(5) merely expands Sono's definition of the independent goods into the case of two goods $x_1$, $x_2$. (For in detail, see Morishima(6), p.94.) It implies that the percentage rates of change of the marginal rate of substitution of goods $x_1$, $x_2$ for all goods in $N_s \cup N_t$ are equal and also only dependent of the quantity of goods $x_1$, $x_2$.

Now, it is easily seen that the following theorem can be directly derived from def. 2 and def. 3.

*Theorem* 1 : In the case where a partition of goods can be reduced to two groups of goods which each must be non-empty, then the independent goods imply the weak separability.

*Proof*: By a simple calculus, we can show that the formula(5) yields the formula(2)

$$0 = \frac{U_k^2}{U_j^2}\left(\frac{U_j}{U_k}\cdot\frac{\partial}{\partial x_k}\left(\frac{U_i}{U_j}\right) - \frac{U_i}{U_k}\cdot\frac{\partial}{\partial x_k}\left(\frac{U_j}{U_k}\right)\right)$$

$$= \frac{U_k^2}{U_j^2}\frac{(U_{ik}U_j - U_iU_{jk})}{U_k^2} = \frac{\partial}{\partial x_k}\left(\frac{U_i}{U_j}\right) \qquad \text{(Q.E. D.)}$$

We have known that if the utility function $U(x)$ fulfilled the condition of def. 3, then $U(x)$ is of the form

$$U(x) = F(u^1(x_1, x_2) + u^2(x_3, x_4, \ldots, x_n)) \qquad \cdots\cdots\cdots (6)$$

where $u^1$ corresponds $U^r$ in the formula(3) and $u^2$ does $U^q$ which is defined on the reunified set of goods $N_q = N_s \cup N_t$. (See Morishima(6), p.95.)

The functional form of those formulae (3) and (6) are called "an additive separable utility function", and the additive separable utility function is equivalent to the strong separability given by def. 1. (See Goldman and Uzawa(2))

But, in Goldman and Uzawa(2), the definition of the strong separability needs more than three goods, because each partition $N_r$, $N_s$, $N_t$ must be non-empty. So, if we would employ the assumption of the independent goods on the consumers preference, even in the case of two goods, we are able to use the additive separable utility function for economic analyses.

In the next section, we shall define the generalized concept of independent goods with respect to a partition ($N_r$, $N_s$, $N_t$) and make it clear that the generalized independent goods is equivalent to the strong separability.

2. WE SHALL GENERALIZE Sono's notion of the independent goods as follows.

*Definition* 4: Let ($N_r$, $N_s$. $N_t$) be a partition of the set $N$. Then, these subsets $N_r$, $N_s$, $N_t$ are called "independent with one another" if the utility function $U(x)$ fulfilled the following conditions with respect to a partition ($N_r$, $N_s$, $N_t$) such that

$$\frac{\frac{\partial}{\partial x_i}\left(\frac{U_j}{U_l}\right)}{\frac{U_j}{U_l}} = \frac{\frac{\partial}{\partial x_i}\left(\frac{U_k}{U_l}\right)}{\frac{U_k}{U_l}} = \theta_i^l(x^r), \text{ for all } i,\ l \in N_r, j,\ k \in N_s \cup N_t, j \neq k. \quad (7)$$

$$\frac{\frac{\partial}{\partial x_j}\left(\frac{U_k}{U_m}\right)}{\frac{U_k}{U_m}} = \frac{\frac{\partial}{\partial x_j}\left(\frac{U_i}{U_m}\right)}{\frac{U_i}{U_m}} = \theta_j^m(x^s), \text{ for all } j,\ m \in N_s,\ i,\ k \in N_t \cup N_r,\ k \neq i \quad \cdots\cdots\cdots\cdots(8)$$

$$\frac{\frac{\partial}{\partial x_k}\left(\frac{U_i}{U_p}\right)}{\frac{U_i}{U_p}} = \frac{\frac{\partial}{\partial x_k}\left(\frac{U_j}{U_p}\right)}{\frac{U_j}{U_p}} = \theta_k^p(x_t), \text{for all } k,\ p \in N_t, i,\ j \in N_s \cup N_r,\ i \neq j \quad (9)$$

where $\theta_i^l(x^r)$ is a function of a subvector $x^r = (x_i)$, $i \in N_r$. $\theta_j^m(x^s)$ and $\theta_k^q(x^t)$ also have a same sort of characteristic of $\theta_i^l(x^r)$.

The formula(7) implies that the percentage rate of change of the marginal rate of substitution of good $x_1$ in $N_r$ for $x_j$ and $x_k$ in $N_s \cup N_t$ are equal with respect to the change of quantity of $x_i$ and only dependent on the quantity of components of the subvector $x^r$. The same statement can be said about each group of goods $N_s$ and $N_t$. Therefore, as we have seen, these formulae (7), (8) and (9) naturally expand Sono's notion of the independent goods. Thus, we have a following theorem.

*Theorem* 2: Let $(N_r, N_s, N_t)$ be a partition of the set $N$. These non-empty subsets $N_r$, $N_s$ and $N_t$ are independent with one another if, and only if, $U(x)$ is strongly separable with respect to a partition $(N_r, N_s, N_t)$.

*Proof*: From the definition 1, we have

$$\frac{\partial}{\partial x_k}\left(\frac{U_i}{U_j}\right) = 0, \text{ for all } i \in N_r,\ j \in N_s,\ k \in N_t$$

Therefore, we can get the following calculus such that

$$\begin{aligned}\frac{\partial}{\partial x_k}\left(\frac{U_i}{U_j}\right) &= \frac{U_p^2}{U_j^2}\cdot\frac{U_{ik}U_j - U_iU_{jk}}{U_p^2} = \frac{U_p^2}{U_j^2}\cdot\left(\frac{U_p(U_{ik}U_j - U_iU_{jk})}{U_p^3}\right)\\ &= \frac{U_p^2}{U_j^2}\left(\frac{U_j}{U_p}\left(\frac{U_{ik}U_{pt} - U_iU_{pk}}{U_p^2}\right) - \frac{U_i}{U_p}\left(\frac{U_{jk}\,U_p - U_jU_{pk}}{U_p^2}\right)\right)\\ &= \frac{U_p^2}{U_j^2}\left(\frac{U_j}{U_p}\cdot\frac{\partial}{\partial x_k}\left(\frac{U_i}{U_p}\right) - \frac{U_i}{U_p}\cdot\frac{\partial}{\partial x_k}\left(\frac{U_j}{U_p}\right)\right) = 0 \quad \ldots\ldots(10)\end{aligned}$$

Obviously, the formula(10) is mathematically equivalent to the left-hand side of the formula(9). By exchanging indices of other group of goods, we can show that the formula(1) yields the left-hand side of formulas(7) and (8). Furthermore, it is self-evident from the definition of the strong separability that the left-hand side of the formula(9) is only dependent on

$x^t$ in $N_t$. (Q.E.D.)

3. THE STUDIES ON THE SEPARABILITY have been based on the purpose in giving an answer to the aggregation problem. In this section, we shall sum up the main point of the relationship between the independent goods and the aggregation problem. We shall begin to make a note of the aggregation problem in the case of macro-production function.

The macro-production function of an economy shall be given the following form;

$$Y = F(X, Z) \qquad \cdots\cdots \quad (11)$$

where $Y$ is a total or social output level, $X$ is a total capital input level and $Z$ is a total labor input level.

Let's assume an economy in which there are $m$ numbers firms. The $i$-th firm produces one product by inputting $x_i$ as capital goods and $z_i$ as labor services. The production function of the $i$-th firm is represented as follow;

$$y_i = f^i(x_i, z_i) = f^i(x_{i1}, x_{i2}, \ldots, x_{ir}, z_{i1}, \ldots, z_{it}) \qquad \cdots\cdots \quad (12)$$

where $x_i$ and $z_i$ are vectors. That is to say, the $i$-th firm is to be assumed to employ $r$ heterogeneous capital goods and $t$ heterogeneous labor services.

Then, the aggregation problem is a problem that what conditions ensure us to make a social output level $Y$ and input level $X$ and $Z$ from the individual firm's input-output turple $(y_i, x_i, z_i)$, when we have the idea that the macro-parameter $Y$, $X$ and $Z$ are based on the decision making of each individual firm.

A solution for the problem is to assume that the aggregation functions exist and are of the form.

$$X = \Psi^1(x_{11}, x_{12}, \ldots, x_{1r}, x_{21}, x_{22}, \ldots, x_{m1}, \ldots, x_{mr})$$

$$Y = \Psi^2(y_1, y_2, \ldots, y_i, \ldots, y_m) \qquad \cdots\cdots\cdots\cdots \quad (13)$$

$$Z = \Psi^3(z_{11}, z_{12}, \ldots, z_{1t}, z_{21}, z_{22}, \ldots, z_{m1}, \ldots, z_{mt})$$

Then Morishima(6) refined some characteristics which those aggregation functions $\Psi^1$, $\Psi^2$ and $\Psi^3$ should be satisfied. These characteristics are to be in below forms.

$$\frac{dx_{ik}}{dx_{ij}} = -\frac{\dfrac{\partial\Psi^1}{\partial x_{ij}}}{\dfrac{\partial\Psi^1}{\partial x_{ik}}} \quad \text{for all } i\in(1, \ldots, m),\ k, j\in(1, \ldots, r) \qquad \cdots\cdots(14)$$

$$\frac{dz_{ih}}{dz_i^l} = -\frac{\dfrac{\partial\Psi^3}{\partial z_{il}}}{\dfrac{\partial\Psi^3}{\partial z_{ih}}} \quad \text{for all } i\in(1, \ldots, m),\ h, l\in(1, \ldots, t) \qquad \cdots\cdots(15)$$

$$\frac{\dfrac{dy_i}{dz_{il}}}{\dfrac{dy_f}{d_{fh}}} = \frac{\dfrac{\partial\Psi^3}{\partial z_{il}}}{\dfrac{\partial\Psi^3}{\partial z_{fh}}} \cdot \frac{\dfrac{\partial\Psi^2}{\partial y_f}}{\dfrac{\partial\Psi^2}{\partial y_i}} \quad \begin{array}{l}\text{for all } i, f\in(1, \ldots, m)\\ \text{all } h, l\in(1, \ldots, t)\end{array} \qquad \cdots\cdots(16)$$

$$\frac{\dfrac{dy_i}{dx_{ij}}}{\dfrac{dy_f}{dx_{fk}}} = \frac{\dfrac{\partial\Psi^1}{\partial x_{ij}}}{\dfrac{\partial\Psi^1}{\partial x_{fk}}} \cdot \frac{\dfrac{\partial\Psi^2}{\partial y_f}}{\dfrac{\partial\Psi^2}{\partial y_i}} \quad \begin{array}{l}\text{for all } i, f\in(1, \ldots, m)\\ \text{all } j, k\in(1, \ldots, r)\end{array} \qquad \cdots\cdots(17)$$

These above formulae(14)–(17) are called "Klein's conditions" by Morishima(6).(1) These imply that the formula(14) is a condition which the quantity of the capital good $X$ does not change when a firm substitutes a capital good $X_{ik}$ for $x_{ij}$. The formula(15) has a same sort for labor inputs. The formula(16) implies that a social output level $Y$ does not change when a labor services are transferred from $i$-th firm to $f$-th firm, provied that $Z$ is constant.(17) is in the case of capital goods.

It has been an assertion that these conditions are necessary for the possibility of the construction of macro-parameter $X$, $Y$ and $Z$ from micro-

(1) The following contexts in this section are essentially depend on Morishima (6).

variables $x_i$, $y_i$ and $z_i$. But, these Klein's conditions implicitly assume that the production function of $i$-th firm satisfies the weak separability *w.r.t* a partition of inputs $(x_i, z_i)$. We can briefly confirm this point. The formula (13) tells us that the right-hand side of the formula(14) is a function of variables $(x_{11}, x_{12}, \ldots, x_{1r}, x_{21}, \ldots, x_{2r}, \ldots, x_{m1}, \ldots, x_{mr})$. The left-hand side of (14) represents a marginal rate of substitution of inputs $k$ and $j$ which is a function of $(x_{i1}, x_{i2}, \ldots, x_{ir})$. Therefore, we know a following characteristic of the production function of the $i$-th firm.

$$\frac{\partial}{\partial z_{il}}\left(\frac{\dfrac{\partial f^i}{\partial x_{ij}}}{\dfrac{\partial f^i}{\partial x_{ik}}}\right)=0 \qquad \begin{array}{l}\text{for all } i\in(1, \ldots, m)\\ \text{all } j,\ k\in(1, \ldots, r)\\ \text{all } l\in(1, \ldots, t)\end{array} \qquad \ldots\ldots\ldots(18)$$

Similarly, we can get the below from the formulae(13) and (15).

$$\frac{\partial}{\partial x_{ij}}\left(\frac{\dfrac{\partial f^i}{\partial z_{il}}}{\dfrac{\partial f^i}{\partial z_{ih}}}\right)=0 \qquad \begin{array}{l}\text{for all } h,\ l\in(1, \ldots, t)\\ \text{all } i\in(1, \ldots, m)\\ \text{all } j\in(1, \ldots, r)\end{array} \qquad \ldots\ldots\ldots(19)$$

These characteristics (18) and (19) are equivalent to the weak separability of the production function $f^i$ *w.r.t* a partition of capital goods and labor services. Namely, we are able to have a corollary from the above consideration and theorem 1 in section 2.

*Corollary*: The independency with respect to a partition of capital goods and lavor services is necessary for the Klein's condition.[2]

The studies on the theory of groups of commodities in Hicks(4) also have an aspect of the aggregation problem. In close of this section, we shall point out the independency in the theory of groups of commodities. Let's

(2) The term of the independence represents that any capital good and labor service are independent goods *w.r.t* a partition of $(x_i, z_i)$.

quote the production function (12) and omit the firm index $i$ for simplicity.

$$y = f(x_1, x_2, \ldots, x_r, z_1, z_2, \ldots, z_t) \qquad \ldots\ldots(20)$$

Hicks(4) had argued the possibility to form a pseudo-good of the group of heterogeneous goods. That is to say, in our framework, the main part of Hicks' argument is to assume the existence of the function such that

$$\xi_1 = \xi_1(x_1, x_2, \ldots, x_i, \ldots, x_r)$$
$$\xi_2 = \xi_2(z_1, z_2, \ldots, z_j, \ldots, z_t)$$

where $\xi_1$ and $\xi_2$ stand for the quantity of pseudo-good. Then, the production function may be reduced to the below form.

$$y = f(x_1, \ldots, x_r, z_1, \ldots, z_t) = F(\xi_1, \xi_2)$$

In Morishima (6), some assumptions which enable us to consider the pseudo-good have been prepared. The mains of those assumptions are;

[A–1]: The production function is weakly separable *w.r.t* a partition of $(x_1 \ldots, x_r)$ and $(z_1 \ldots, z_t)$.

[A–2]:

$$\frac{\partial}{\partial x_1}\left(\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_1}}\right) = 0 \quad \text{for all } i \in (2, 3, \ldots, r), \text{ and}$$

$$\frac{\partial}{\partial z_j}\left(\frac{\frac{\partial f}{\partial z_j}}{\frac{\partial f}{\partial z_1}}\right) = 0 \quad \text{for all } j \in (2, 3, \ldots, t)$$

Let's consider a following formula.

$$dx_1 + \sum_{k=2}^{r}\left(\frac{\frac{\partial f}{\partial x_k}}{\frac{\partial f}{\partial x_1}}\right) dx_k \qquad \ldots\ldots(21)$$

The formula (21) is a normalization of an exchange of output level derived from a small exchange of capital goods in term of the marginal productivity of good 1. Then, it is obvious from the weak separability assumption [A–1] that the formula (21) is only dependent on $x=(x_1, \dots, x_r)$. Next, we can get the formula (22) from [A–2].(3)

$$\frac{\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)}{\frac{\partial f}{\partial x_1}}=\frac{\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_i}\right)}{\frac{\partial f}{\partial x_i}} \quad \text{for all } i \in (2, \dots, r) \qquad \dots\dots\dots\dots(22)$$

Therefore, we have the below.

$$\frac{\partial f}{\partial x_i}\cdot\frac{\partial^2 f}{\partial x_1 \partial x_j}=\frac{\partial f}{\partial x_j}\cdot\frac{\partial^2 f}{\partial x_1 \partial x_i} \quad \text{for all } i \neq j \qquad \dots\dots\dots\dots(23)$$

Because the formula (21) is of the form of total difference and the formula (23) ensures us the integrability of it, we are able to get the following form.

$$\int\left(dx_1+\sum\frac{\frac{\partial f}{\partial x_k}}{\frac{\partial f}{\partial x_1}}\cdot dx_k\right)=\xi_1(x_1, x_2, \dots, x_r)$$

That is to say, we can regard $\xi_1$ as a quantity of pseudo-good and apply the same discussion to the case of labor services $z=(z_1, \dots, z_t)$.

In the above consideration, the weak separability has been in an important role. In other words, the independence *w. r. t* a partition $x$ and $z$ is to be very significant.

(3) We can get the formula (22) straightforwardly by a simple calculus, i.e.

$$\frac{\partial}{\partial x_1}\left(\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_1}}\right)=\frac{\frac{\partial f}{\partial x_1}\cdot\frac{\partial^2 f}{\partial x_i \partial x_1}-\frac{\partial f}{\partial x_i}\cdot\frac{\partial^2 f}{\partial x_1 \partial x_1}}{\frac{\partial f}{\partial x_1}}=0$$

4. IN THE ABOVE SECTIONS, it is showed that the independent goods is a sufficient condition for the weak and strong separability. The additive separable function have been an important role in many fields of economic theory. For instance, Negishi (7) showed that a sufficient condition for the neutrality of money was that an individual utility function should be an additive separable with respect to a partition of monoy and other goods.

Similarly, in the context of the aggregation problem, Klein's condition implicitly contains an assumption of the additive separability of production function.

These contexts inform us that if we would use an additive separable utility function to research into some economic problem, then, even in the case of two goods, it is enough for us to assume the independence condition with respect to a partition of commodities.

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