Reliability assessment of project duration using method of moments

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CONTENTS

モーメント法に基づくプロジェクト納期の信頼性評価

プロジェクト納期の信頼性評価はプロジェクトの管理に対して非常（に重要な部分である。これを踏まえて、人々はプロジェクトの納期が目標納期達成する可能性を評価するため、以前の研究者は効率的な評価方法を多く提出されました。

プログラム・エバリューション・アンド・レビュー・テクニック（PERT）は通常、プロジェクトの納期達成する可能性を評価するために、よく使われている方法となる。PERT に基づいて、作成されたプロジェクト納期の信頼性解析に関する既存の研究はプロジェクト納期が正規分布に従うことを仮定されていますが、実際のプロジェクトは納期の分布は不確実である。上記の問題を解決するため、立方正規変換に基づくプロジェクト納期の信頼性解析方法提案しました。提案手法では、最初の 4 次モーメント（平均、標準偏差、歪度、尖度）によって決定される立方正規分布を採用して、プロジェクト納期の分布に適合させました。 その立方正規分布を取得した後、プロジェクト納期のより合理的な信頼性分析が実施されました。 実用的な工学的事例を通して、提案された方法がより正確で効率的であることが証明された。

これまでのほとんどの研究では、プロジェクト納期の信頼性を評価する際に 1 つのキーパスしか考慮されていませんでした。しかし、研究によって、パス間に相関関係があるので、プロジェクト納期の信頼性に対して大きな影響を与えパス述べていました。プロジェクト納期の信頼性を評価する際にキーパスのみを考慮されていることは不合理であるので、研究者はすべてのネットワークパスの影響を考慮し、プロジェクト納期の信頼性を評価するために、多く方法を提案されている。ただし、既存の方法は、各パス間の相互相関係数と各パスの同時破壊確率を計算する必要がある。煩雑な計算手順を行わなくてはならないと思いる。既存法の問題を解決するため、プロジェクト納期の信頼性を評価するために、モーメント法に基づく簡単で効率的な方法が提案されている。計算例の結果から、提案法はモンテカルロシミュレーション法と同じ精度の結果が得られるが、計算量が少ないである。また、提案法は既存法と比較して、各パス間の相関係数を計算する必要がなくて、各パス間の同時破壊確率を計算する必要はありません。

Abstract

The reliability assessment of the project duration is one of the most important parts in the process of the project management, which has prompted many effective methods to be developed to assess the possibility of project duration meeting the target duration.

The program evaluation and review technique (PERT) is usually used to assess the reliability of the project duration. Most of the existing studies on the reliability analysis of the project duration prepared by PERT assumed that the project duration is subject to a normal distribution. However, in actual projects, the distribution of the total project duration is usually unknown. To address this issue, a flexible distribution is utilized to represent the distribution of the total construction duration. In the proposed method, the cubic normal distribution which determined by its first four moments (mean, standard deviation, skewness, and kurtosis) was adopted to fit the distribution of the total construction duration. After obtaining its cubic normal distribution, a more rational reliability analysis of the total construction duration was conducted. Through a practical engineering case, it is proved that the proposed method is more accurate and efficient.

Most previous studies have only considered one critical path when evaluating the reliability of the project duration. And the evaluation results yielded by these methods are often inaccurate; mainly because they disregard the correlation between network paths, which has been proven to significantly affects the reliability of the project duration. Therefore, to consider the influence of all network paths, many methods have been proposed to evaluate the reliability of project duration. However, existing methods require finding the representative paths and complex calculations of mutual correlation coefficients between each pair of paths and the joint failure probability of each pair of representative paths. In this thesis, efficient and effective methods for assessing the reliability of project duration based on the method of moments is proposed, which utilizes bivariate-dimension reduction technology, third-order moment reliability index and fourth-moment transformation. The proposed methods can overcome the extensive calculations of various factors in the commonly used methods: the correlation coefficients between each pair of paths and the joint failure probability of each pair of representative paths. From numerical examples, it can be concluded that the proposed method can achieve nearly the same results as Monte Carlo simulations with less calculation.

CHAPTER 1

Introduction

1.1 Background

The reliability assessment of the project duration is one of the most important parts in the process of the project management, especially for those projects which consider project time as targets. Delivering project on-time or not has a lot to do with the profits of the participants. This has prompted many effective methods to be developed to assess the possibility of project duration meeting the target duration.

The critical path method (CPM) has been popularly used to assess project duration for decades [1-4]. The CPM assumes each activity duration of the project as a deterministic value, while in practice, the duration of each activity in a project is affected by many external factors (like weather, site condition and productivity) and is uncertain [5-6]. To deal with this problem, Program Evaluation and Review Technique (PERT) [7-8] was developed in 1959. The activity duration in PERT was regarded as a random variable, and the three-time estimation was used to establish an estimation model of project activity duration. Since then, it has become a common tool for reliability assessment of project duration. Most of the existing studies on the reliability analysis of the project duration prepared by PERT assumed that the project duration is subject to a normal distribution. However, in actual projects, the distribution of the total project duration is usually unknown.

Another important issue is that most previous studies have only considered one critical path when evaluating the reliability of the project duration. And the evaluation results yielded by these methods are often inaccurate [9-12]; mainly because they disregard the correlation between network paths, which has been proven to significantly affects the reliability of the project duration [4,13]. Therefore, researchers have recently focused on considering the impact of all the network paths when assessing the reliability of the project duration. Several probabilistic scheduling methods, such as narrow reliability bounds (NRB) [13-15], the Monte Carlo simulation (MCS) [16-21], and simplified Monte Carlo simulation (SMCS) [10], have been proposed. However, existing methods require finding the representative paths and complex calculations of mutual correlation coefficients between each pair of paths and the joint failure probability of each pair of representative paths.

Meanwhile, in traditional methods for reliability evaluation of project duration, it is assumed that the durations of individual activities are independent. In real situations, however, factors such as weather, site conditions, and design changes can affect the duration of project activities. These factors usually affect multiple activities on a particular project and may cause the activity duration to be correlated [22]. Traditional methods will not capture the correlation that may exist between the durations of different activities in a project network.

Therefore, accuracy and effective methods for reliability assessment of project duration must be developed.

1.2 Objective

This study aims to solve three main problems:

Firstly, the shortcomings of the existing Program Evaluation and Review Technique (PERT) will be solved. A flexible distribution would be utilized to represent the distribution of the total construction duration. In the proposed method, the cubic normal distribution which determined by its first four moments (mean, standard deviation, skewness, and kurtosis) will be adopted to fit the distribution of the total construction duration. After obtaining its cubic normal distribution, a more rational reliability analysis of the total construction duration can be conducted. And the proposed method will be more accurate and efficient than PERT.

Secondly, considering the impact of all the network paths a simple and effective method for evaluating the reliability of the project duration based on the method of moments will be proposed. The proposed method does not have to assume the distribution of the project duration and calculate the mutual correlation coefficients between each pair of paths and the joint failure probability of each pair of representative paths.

Thirdly, a Monte Carlo Simulation (MCS) based on fourth-moment transformation technique for reliability assessment method of project duration will be proposed to deal with the correlation problem that may exist between the durations of different activities in a project network.

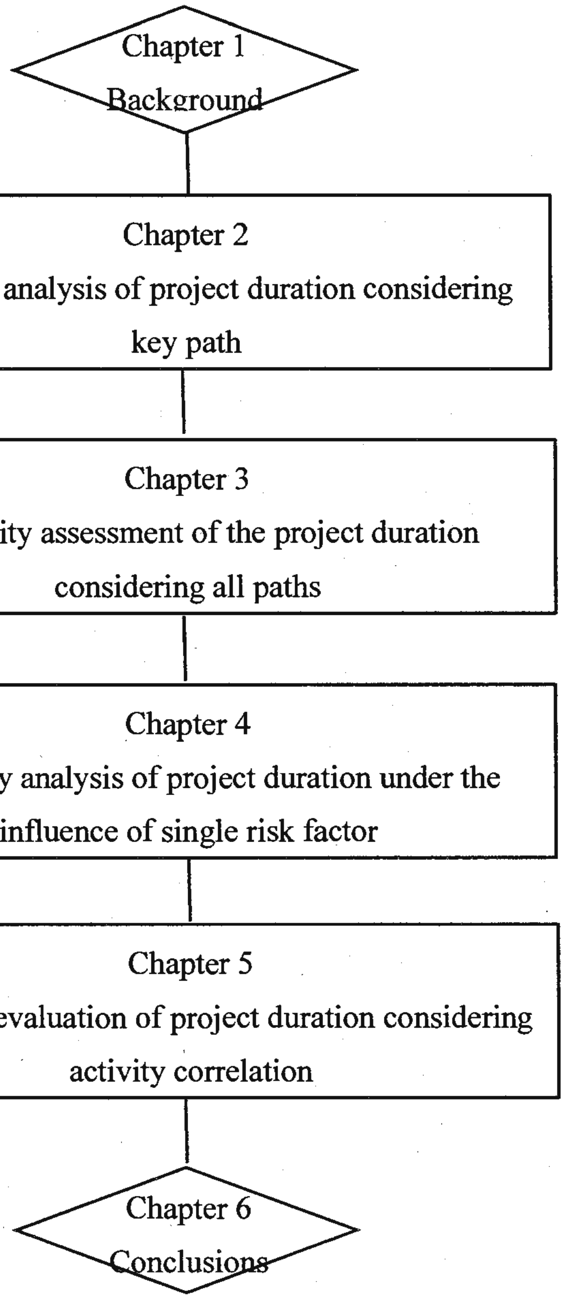
1.3 Organization

The background was introduced in Chapter 1.

The thesis consisted of three parts: (1) reliability analysis of project duration considering key path (Chapter 2), (2) reliability assessment of the project duration considering all paths (Chapter 3), (3) reliability evaluation of project duration considering activity correlation (Chapter 4). In Chapter 2, the cubic normal distribution which determined by its first four moments (mean, standard deviation, skewness, and kurtosis) was utilized to represent the distribution of the total construction duration.

In Chapter 3, an efficient and effective method for assessing the reliability of project duration based on the method of moments is proposed, which utilizes bivariate-dimension reduction technology and third-order moment reliability index. In Chapter 4, a Monte Carlo simulation (MCS) based on fourth-moment transformation technique for reliability assessment method of project duration is proposed.

In the last Chapter, the significant innovations in Chapter 2 to 4 were summarized.



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List of publications

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[1] **Lu Ren**, Haizhong Zhang, Pei-Pei Li and Yan-Gang Zhao. Reliability evaluation of project completion time using fourth-moment normal transformation. *J. Struct. Engrg. AIJ*, 2021, 67B. (accepted)

[2] **Lu Ren,** Pei-Pei Li and Yan-Gang Zhao. An efficient and effective method for reliability assessment of project. *Quality Technology & Quantitative Management*, 2020. (Under review)

International conference paper

[3] **Lu Ren,** Haizhong Zhang, Pei-Pei Li and Yan-Gang Zhao: Reliability assessment of project duration based on cubic normal transformation, *The International Engineering Mechanics Forum* (*IEMForum 2020*), Feb. 22-25, 2021, Yokohama, Japan.

[4] **Lu Ren** and Yan-Gang Zhao: Reliability analysis of total activity duration in PERT using cubic normal distribution, *7th International Symposium on Reliability Engineering and Risk Management* (*ISRERM2020*), Nov. 12-14, 2020, Beijing, China.

[5] **Lu Ren** and Yan-Gang Zhao: Reliability analysis of total construction duration based on cubic normal distribution, *International Symposium on Reliability of Multi-disciplinary Engineering Systems under Uncertainty* (*ISRMES2019*), Dec. 8-11, 2019, Taipei, China.

[6] **Lu Ren** and Yan-Gang Zhao: Reliability analysis of project duration plan based on BIM, *21th International Conference on Application of Probability and Statistics in Civil Engineering* (*ICAPSCE 2019*), Mar. 14-15, 2019, London, United Kingdom.

Domestic conference paper

[7] **Lu. Ren** and Yan-Gang Zhao：A simple method to evaluate reliability of the project completion time, *J. Struct. Constr. Engrg. AIJ*, 2020, 13-14.

[8] **Lu Ren** and Yan-Gang Zhao : Reliability analysis of project duration plan based on cubic normal transformation, *J. Struct. Constr. Engrg. AIJ*, 2019, 63-64.

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CHAPTER 2

Reliability analysis of project duration considering key path

2.1 Introduction

Building Information Modeling (BIM) is a recognized advanced technology that enhances the effectiveness and efficiency of projects in the construction, engineering, and construction industries [1]. For the project, the total construction duration is directly linked to the cost, so that the evaluation of the reliability of it is particularly important for the entire project. Program evaluation and review technique (PERT) is one of the most commonly and classically used tools in project management of project duration plan since its appearance in 1959 [2-3]. It assumed that the total construction duration is subject to a normal distribution to evaluate its reliability [4]. However, in actual, the distribution of the total construction duration is uncertain. Therefore, the reliability prediction result given by PERT is not reasonable. This chapter utilized cubic normal distribution to represent the distribution of the total construction duration. The proposed method is as follows: firstly, building the BIM model of the project and integrating the original construction duration with BIM model; secondly, formulating the reliability model of total construction duration; thirdly, using cubic normal distribution (determined by its first four moments, mean, standard deviation, skewness, and kurtosis) to fit the distribution of the total construction duration and conducting the reliability analysis of it. After obtaining its four-parameter distribution, a more rational reliability analysis of the total construction duration was conducted. Through case analysis, it is proved that this method is more accurate and efficient.

2.2 Formulating the reliability model of total construction duration

Due to the characteristics of the construction project, it should guarantee the reliability of the construction duration, the possibility of safe construction, the controllable degree of cost and the quality required to meet the requirements of the specification [5-8]. In order to assess the reliability of the project duration, an effective reliability analysis model is proposed in this section.

2.2.1 Assumptions of the reliability model

Based on the existing design information, establish a BIM model, integrate it with project durations. The project duration has its own specific calculation principles and methods, usually obtained using PERT. The PERT model in this paper was developed using assumptions similarly to the traditional PERT method, as follows:

Assumption 1

The probability of the duration of the activity is using the beta distribution to represent, which can be determined using three times estimate: the most optimistic time a, the most pessimistic time b and the most probable time m. The optimistic time a and pessimistic time b represent the extreme values of the probability distribution (ie 0 and 100%), the most likely time is m. Studies have shown that the duration of many construction activities can be expressed in terms of beta distribution [9].

The beta distribution is defined over the (0,1) interval, through the affine transformation Y= a + (b − a) X allows the beta distribution to be defined in any finite interval (a, b). The original PERT obtains the calculation formula of the mean and variance of the construction duration by assuming the parameters of the beta distribution p=4, q=4. However, in practice this assumption does not necessarily hold. Therefore, in order to select a more reasonable beta distribution assumed that the probability of m is k times of the probability of a and b. Then let $pq = k^2$ and $y_0 = m$, so that by solving the equations $pq = k^2$ and $(p-1)/(p+q-2) = (m-a)/(b-a)$, the values of p and q can be obtained.

Assumption 2

The activities in the network path are not correlated, the total construction duration is determined by the key route of the project, which takes the longest time, as shown in Figure 2-1.

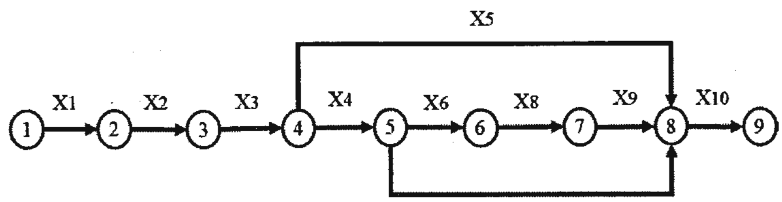

Figure 2-1. The arrow diagram

2.2.2 Reliability model of total construction duration

The performance function G of construction duration is defined as follows:

$$G(X) = T_M - \sum_{i=1}^{n} x_i \tag{2-1}$$

where T_M indicates the prescribed total planned duration, x_i represents the duration of each sub-item of the critical path. The formulas for calculating the first four moments of the total construction duration are:

$$\mu_G = \sum_{i=1}^{n} \mu_i \tag{2-2}$$

$$\sigma_G^2 = \sum_{i=1}^{n} \sigma_i^2 \tag{2-3}$$

$$\alpha_{3G}\sigma_G{}^3 = \sum_{i=1}^{n} \alpha_{3i}\sigma_i^3 \tag{2-4}$$

$$\alpha_{4G}\sigma_G{}^4 = \sum_{i=1}^{n} \alpha_{4i}\sigma_i^4 + 6\sum_{i=1}^{n-1}\sum_{j>i}^{n} \sigma_i^2\sigma_j^2 \tag{2-5}$$

The reliability analysis of the first four moments of the total construction period is carried out by using the cubic normal distribution.

2.3 Reliability analysis of total construction duration

In the existing model that the distributions of random variables in project duration are assumed to follow normal distribution. It is inappropriate to use normal distribution to deal with the random variables in project duration, cause the distributions of random variables are generally unknown, and in most cases, they do not obey normal distributions. While it is easy to obtain their first four moments so that the cubic normal distribution is used in this study, which has a single expression and can more effectively fit the histograms of available data than normal distribution [10].

For a random variable G, if its first four moments mean (μ_G), standard deviation (σ_G), skewness (α_G), and kurtosis (α_{4G}) are known, the standardized random variable G_s can be expressed by the cubic normal transformation: $G_s = (G-\mu_G) / \sigma_G$, as follows [11]:

$$G_s = S_u(u, \mathbf{M}) = a_1 + a_2 u + a_3 u^2 + a_4 u^3 \tag{2-6}$$

in which, G_s is a standardized random variable, with the skewness and kurtosis are the same as those of G, $\mathbf{M}$ is a vector denoting the first four moments of G, $S_u(u)$ is a third-order polynomial of u, Φ and ϕ are the cumulative distribution function (CDF) and probability density function (PDF) of a standard normal random variable u; a_1, a_2, a_3 and a_4 are coefficients up to the first four moments of the left side of Eq. (6) and it is equal to those of the right side [12]. Therefore, the inverse function of the relationship between standard normal variable u and the standardized variable G_s can then be expressed as

$$u = S^{-1}(G_s, \mathbf{M}) \tag{2-7}$$

where the S^{-1} denotes inverse function of S and the explicit expression of u are summarized in Table 2-1 [13].

The parameters p, q, Δ, θ, J_1^*, J_2^* and J_0 of Table 1 can be given as follows:

$$p = \frac{3b - a^2}{9} \tag{2-8}$$

$$q = \frac{a^2}{27} - \frac{ab}{6} - \frac{a}{2} - \frac{G_s}{2a_4}, \Delta = \sqrt{p^3 + q^2} \tag{2-9}$$

$$\theta = \arccos\left[-q / (\sqrt{-p})^3 \right] \tag{2-10}$$

$$J_1^* = \sigma_G a_4 (-2|p|^{3/2} + 2q + G_s / a_4) + \mu_G \tag{2-11a}$$

$$J_2^* = \sigma_G a_4 (2|p|^{3/2} + 2q + G_s / a_4) + \mu_G \tag{2-11b}$$

$$J_0 = -(a_2^2 / 4a_3 + a_3)\delta_G + \mu_G \tag{2-12}$$

where parameters a and b are respectively defined as $a = a_3/a_4$ and $b = a_2/a_4$. It can be seen from Table 2-1 that there are six types in the cubic normal distribution, including unbounded distributions (Types I and VI), unilaterally bounded distributions (Types II, III, and V), and a bounded distribution (Type IV).

Table 2-1 Expressions of u

Parameters			u	Range of G	Type
$p \geq 0$			$-p / \sqrt[3]{\Delta - p} + \sqrt[3]{\Delta - p} - a/3$	$(-\infty, \infty)$	I
		$\alpha_{3G} \geq 0$	$2\sqrt{-p}\cos(\theta/3) - a/3$	$J_1^* < G < J_2^*$	II
	$a_4 > 0$	$\alpha_{3G} < 0$	$-p / \sqrt[3]{\Delta - p} + \sqrt[3]{\Delta - p} - a/3$	$G \geq J_2^*$	
			$-p / \sqrt{\Delta - p} + \sqrt{\Delta - p} - a/3$	$G \leq J_1^*$	III
			$-p / \sqrt{-p}\cos[(\theta - \pi)/3] - a/3$	$J_1^* < G < J_2^*$	
	$a_4 < 0$		$-p / \sqrt{-p}\cos[(\theta + \pi)/3] - a/3$	$J_1^* \leq G \leq J_2^*$	IV
$p < 0$		$\alpha_{3G} > 0$	$\sqrt{1/4 + (a_3/a_2)^2 + a_3 G_s / a_2} - 1/2$	$G \geq J_0$	V
	$a_4 = 0$	$\alpha_{3G} < 0$	$\sqrt{1/4 + (a_3/a_2)^2 + a_3 G_s / a_2} - 1/2$	$G \leq J_0$	
		$\alpha_{3G} = 0$	G_s	$(-\infty, \infty)$	VI

According to Eq. (3-6) the CDF and PDF are expressed as [14]:

$$F(G) = \Phi(u) \tag{2-13}$$

$$f(G) = \frac{\phi(u)}{\sigma_G (3a_4 u^2 + 2a_3 u + a_2)} \tag{2-14}$$

Substituting Eq. (6) with an explicit expression listed in Table. 2-1 into Eq. (2-13), one obtains that

$$F_{G_s}(G_s) = \Phi(u) = \Phi[S^{-1}(G_s, \mathbf{M})] \tag{2-15}$$

Therefore, the failure probability of the total project duration can be expressed as

$$P_F = F_{G_s}(\frac{-\mu_G}{\sigma_G}) = \Phi[S^{-1}(\frac{-\mu_G}{\sigma_G}, \mathbf{M})] \tag{2-16}$$

2.4 Application of the proposed method

Build the BIM model of the project, showing in Figure 2-2. Integrate the project duration information with BIM model. Take a simple but representative part of the project as an example. The prescribed total planned duration of this construction stage is T_M =40 days. It is known that the key path of a sub-item project consists of four parts. The three times estimates data of each activity given by the expert is showing in Table 2-2.

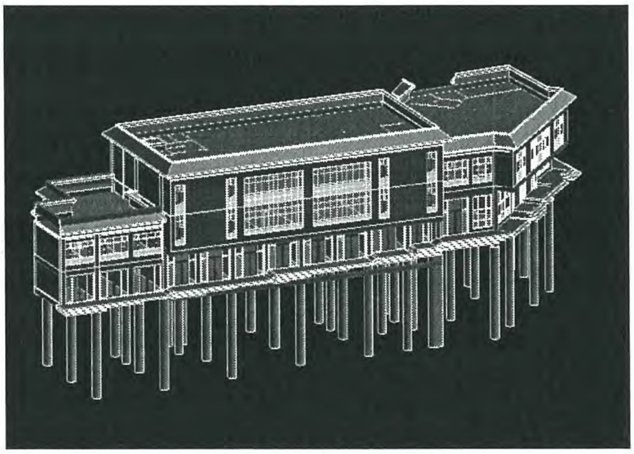

Fig. 2-2. BIM model of the project

Table 2-2 Three time estimates of each activity

Symbol	Duration / d		
	a	m	b
x_1	6	8	14
x_2	5	6	13
x_3	6	8	20
x_4	3	4	11

The performance function G of construction duration is:

$$G(\mathrm{X}) = T_M - \sum_{i=1}^{n} x_i = 40 - \sum_{i=1}^{n} x_i \tag{2-17}$$

When k =1.5, the shape parameters p and q of each sub item can be obtained. And the first four moments of each activity of the project can be calculated according to the definition, as shown in Table 2-3.

According to the Eqs. (2-2)-(2-5), the first four moments of the total construction duration are obtained, as shown in Table 2-4.

Table 2-3 First four moments of each sub-item

	μ	σ^2	α_3	α_4
x_1	10.4852	2.1594	0.2214	2.1041
x_2	7.88	3.5747	0.4644	2.2887
x_3	11.2243	11.1782	0.4069	2.2341
x_4	5.88	3.5746	0.3290	2.2887

Based on the three times estimation of the sub-projects, the simulation of function performance G was carried out using the Monte Carlo method. The number of times is 10000, and the probability distribution histogram of the total construction duration is obtained, as presented in Figure 2-3.

Table 2-4 The first four moments of the data and coefficients of the cubic normal distribution

First four moments		Coefficients of the cubic normal distribution	
μ_G	4.52449	a_1	0.0516
σ_G	4.15886	a_2	1.0662
α_{3G}	-0.2676	a_3	-0.0516
α_{4G}	2.6402	a_4	-0.0234

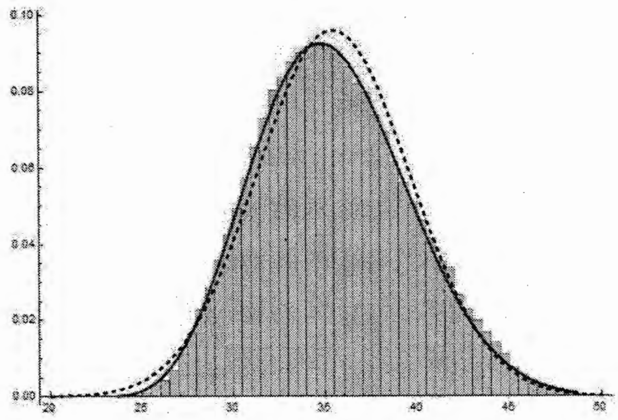

Fig. 2-3 Comparison between normal distribution and cubic normal distribution in fitting statistical data

Figure 2-3 is along with the fitted PDF of the normal distribution, whose mean value and standard deviation are equal to those of the three times estimates data, and the fitted PDF of the cubic normal distribution whose first four moments of G and coefficients are shown in Table 2-4. It can be seen from Figure 2-3 that the cubic normal distribution fits the histogram much better than the normal distribution. According to the cubic normal distribution, the corresponding failure probability of 40 days of the total construction duration is 0.149. It shows that the planned total duration of 40 days is reliable.

2.5 Conclusions

In this Chapter, a new computing model combined with PERT and cubic normal distribution is proposed. It takes advantage of several methods proposed since the original PERT was introduced. A real project was analyzed by comparing cubic normal distribution and normal distribution with Monte Carlo simulation which illustrated that the use of this new proposed model can enable more accurate reliability analysis of the total construction duration for the projects.

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CHAPTER 3

Reliability assessment of the project duration considering all paths

3.1 Introduction

The program evaluation and review technique (PERT) is usually used to assess the reliability of the project duration [1-2]. However, this method often results in inaccurate results and underestimation of the expected project duration [3-6], because only the critical path is considered and the influence of the correlation between the network paths on the overall possibility of the project duration is neglected [7-8]. Studies have proved that the correlation between the network paths is considerably influential.

Therefore, researchers have recently focused on considering the impact of all the network paths when assessing the reliability of the project duration. Several probabilistic scheduling methods, such as narrow reliability bounds (NRB) [8-10], the Monte Carlo simulation (MCS) [11-16], and simplified Monte Carlo simulation (SMCS) [12], have been proposed.

MCS, which is known for its high accuracy, is commonly utilized as a traditional probabilistic scheduling method in the construction industry [12-18] and has become a standard technique used by project managers [19]. SMCS can provide similar results to those obtained using MCS; however, a lower computational load is required [12]. Even though these simulation methods are popular for project management in construction, they can only provide numerical solutions, parametric analysis cannot be conducted.

To solve these problems, some studies tended to use analytical methods to assess the reliability of project duration. However, for large-scale projects, the existing analysis method NRB requires extensive calculations to assess the reliability of the project duration [8].

To overcome the shortcomings in NRB, recently, Li et al. [8] proposed a reliability assessment method named fast and accurate reliability bounds (FARB) to simplify and accelerate the calculation in case of large-scale projects [8]. However, to employ this method, each path of the project network should contain at least five activities; furthermore, the correlation coefficients between each pair of paths and the joint failure probability of each pair of representative paths must be calculated.

Other reliability assessment methods, such as approximation methods [20-23], multivariate methods [24], and fast and accurate risk evaluation (FARE) [7] also exhibit limitations. For example, the approximation method proposed by Ang et al. [20] produces optimistic results, whereas the multivariate methods proposed by Anklesaria and Drezner [24] are only applicable to small networks [8].

To address those shortages described above, this study proposes an efficient and effective method based on the method of moments to assess the reliability of the project duration. In this method, the overall performance function with respect to the project duration is first established. Next, bivariate-dimension reduction is used to evaluate the first three moments of the overall performance function of the project duration. Finally, the reliability of the project duration is estimated using the third-moment reliability index. The efficiency and accuracy of the proposed method are illustrated using two numerical examples compared with the results obtained from the Monte Carlo simulation method.

3.2 Previous review

Considering the influence of all network paths on the reliability assessment of project duration, the project network should be regarded as a series system [25]. For a series system with many paths, the project duration being greater than the target duration is the union of all the possible failure paths. P_F, therefore, is the probability of occurrence of all the possible failure paths and is expressed as

$$P_F = P(f_1 \cup f_2 \cup ... \cup f_L) \tag{3-1}$$

where f_i denotes a possible failure path, indicating that the duration of path i is greater than the target duration; and L denotes the number of paths in the project network.

For a project network, the reliability of the project duration can be estimated using

$$P_S = 1 - P_F \tag{3-2}$$

where P_S denotes the probability that the project can be completed within the specified target duration.

The calculation of P_F in Eq. (3-1) requires the computation of the joint failure probability of path 1 to path L, which is considered as quite complicated. To simplify this calculation, the NRB method proposed by Ditlevsen [9] can be used to express the upper and lower bounds of P_F. According to Eq. (3-2), the upper and lower bounds of P_S can be obtained as follows

$$P_{S_{UB}} = 1 - \left\{ P(f_1) + \sum_{i=2}^{L} \text{Max}\left[0, P(f_i) - \sum_{j=1}^{i-1} P\left(f_i \cap f_j\right) \right] \right\} \tag{3-3a}$$

$$P_{S_{LB}} = 1 - \left\{ P(f_1) + \sum_{i=2}^{L} \left[P(f_i) - \underset{j<i}{\text{Max}}\, P\left(f_i \cap f_j\right) \right] \right\} \tag{3-3b}$$

where the subscripts UB and LB indicate the upper and lower bounds, respectively, and $P(f_i \cap f_j)$ denotes the joint probability that the durations of paths i and j are greater than a target duration. The network paths are sorted from 1 to L according to their decreasing importance; therefore, $P(f_1) > P(f_2) > \ldots > P(f_L)$, $i = 1, \ldots, L$.

However, large-scale projects with many paths require excessive computations because the reliability of the project duration estimated using Eqs. (3-3a)–(3-3b) considers the upper and lower bounds of $P(f_i \cap f_j)$ for any pair of paths in the network [8].

To shorten the computational time of NRB, Li et al. [8] proposed the FARB method to reduce the calculation by judging and truncating the insignificant paths. In this method, the insignificant paths are truncated based on the principle proposed by Ang et al. [8,20]: the paths in the network with high mean durations and high variances have a greater impact on the reliability of project completion time; If the durations on multiple paths are highly correlated, these paths will be replaced by the single representative path with the highest variance in each group of correlated paths; paths with low correlation coefficients are considered independent and are grouped as other representative paths.

According to Ang et al. [20], all the activities are assumed to be statistically independent, the correlation coefficients between paths i and j can be obtained as follows

$$\rho_{ij} = \frac{\sum_z \sigma_{W_z}^2}{\sigma_i \sigma_j} \tag{3-4}$$

where $\sigma_{W_z}^2$ denotes the variance of the duration of each activity W_z; z consists all the activities coexisting in paths i and j; σ_i and σ_j are the standard deviation of the duration of path i and j, respectively.

Then, the correlation coefficient ρ_{ij} of paths i and j is compared with the correlation coefficient $\rho_0 = 0.5$, recommended by Ang et al. [20]. When $\rho_{ij} > \rho_0$, paths i and j are termed as dependent, and the most significant path is selected as a representative path to represent these pairs of highly correlated paths [8]. When $\rho_{ij} < \rho_0$, paths i and j are termed as independent.

If paths i and j are independent and representative paths, the joint probability of any pair of f_i and f_j can be described as

$$P(f_i \cap f_j) = P(f_i)P(f_j) \tag{3-5}$$

By combining Eq. (3-5) with Eqs. (3-3a)–(3-3b), the upper and lower bounds of P_S of the project duration can be obtained as follows

$$P_{S_{UB}} = 1 - \left\{ P(f_1) + \sum_{i=2}^{L} \text{Max}\left[0, P(f_i) - \sum_{j=1}^{i-1} P(f_i)P(f_j) \right] \right\} \tag{3-6a}$$

$$P_{S_{LB}} = 1 - \left\{ P(f_1) + \sum_{i=2}^{L} \left[P(f_i) - \underset{j<i}{\text{Max}}\, P(f_i)P(f_j) \right] \right\} \tag{3-6b}$$

The duration of each path is approximately to a normal distribution [8]. Subsequently, for independent and representative paths, the probability of the project completion time being within the target duration t can be expressed as

$$P_{S_{UB}} = 1 - \left\{ \Phi(\beta_1) + \sum_{i=2}^{L} \text{Max}\left[0, \Phi(\beta_i) - \sum_{j=1}^{i-1} \Phi(\beta_i)\Phi(\beta_j) \right] \right\} \tag{3-7b}$$

$$P_{S_{LB}} = 1 - \left\{ \Phi(\beta_1) + \sum_{i=2}^{L} \left[\Phi(\beta_i) - \underset{j<i}{\text{Max}} \left\{ \Phi(\beta_i)\Phi(\beta_j) \right\} \right] \right\} \tag{3-7b}$$

where $\Phi(\bullet)$ denotes the cumulative distribution function of a standard normal variable $N(0,1)$, and β_i denotes the reliability index of project duration for path i, which can be expressed as [8]

$$\beta_i = \frac{\mu_i - t}{\sigma_i} \tag{3-8}$$

where μ_i denotes the mean of the duration of path i.

To obtain the μ_i and σ_i for each path, the durations of all activities on each path have to be determined. The duration of a project activity is affected by a variety of variable construction risks and uncertainties [26-31], so that it should be regarded as a random variable. Therefore, the PERT method is utilized to determine the mean (μ_{yj}) and variance ($\sigma^2_{y_j}$) of the duration of each activity j, as presented in Eqs. (3-9) and (3-10)

$$\mu_{y_j} = \frac{t_a + 4t_o + t_b}{6} \tag{3-9}$$

$$\sigma^2_{y_j} = \left[\frac{t_b - t_a}{6} \right]^2 \tag{3-10}$$

where t_a, t_b, and t_o denote the optimistic, pessimistic, and most likely activity durations, respectively, empirically estimated by the planner.

However, to use the FARB method, at least five activities are required on each path of the project network. Furthermore, the correlation coefficient between each pair of paths and the joint failure probability of each pair of representative paths need to be computed; these factors are considered troublesome on the current methods of the project duration assessment. In order to successfully deliver the project, the project manager needs to depend on the developed technology to achieve the required efficiency and effectiveness [32-33]. Therefore, an efficient and effective method for reliability assessment of project duration must be developed.

3.3 An efficient and effective reliability assessment method

This section proposes an overall performance function to reduce the computational complexity incurred by the computation of the correlation coefficients between each pair of paths and the joint failure probability between each pair of representative paths. Utilizing the first three moments of the proposed performance function, the reliability of the project duration can be assessed by the third-moment reliability method. The following sections describe the development of the proposed method in more detail.

3.3.1 Establishment of overall performance function of the project duration

For the project network, each failure path f_i can be determined by a performance function $g_i = g_i(\mathbf{X}) = t - T_i(\mathbf{X})$ [34], such that $f_i = (g_i \leq 0)$, and the failure probability of the project duration in Eq. (3-1) can be given as follows

$$\begin{aligned} P_F &= Prob[f_1 \cup f_2 \cup ... \cup f_L] \\ &= Prob[(g_1(\mathbf{X}) \leq 0) \cup (g_2(\mathbf{X}) \leq 0), ..., \cup (g_L(\mathbf{X}) \leq 0)] \end{aligned} \tag{3-11}$$

where t is the target project duration, $\mathbf{X} = \{x_1, x_2, ..., x_n\}$ indicates the vector of random variables, and $T_i(\mathbf{X})$ denotes the duration of path i.

On the contrary, the reliability of the project duration is the probability that none of the L possible failure paths will occur, that is, the intersection of all the complements of the L potential failure paths, which gives

$$\begin{aligned} P_S &= Prob[\overline{f_1 \cup f_2 \cup ... \cup f_L}] = Prob[\overline{f_1} \cap \overline{f_2} \cap ... \cap \overline{f_L}]] \\ &= Prob[(g_1(\mathbf{X}) > 0) \cap (g_2(\mathbf{X}) > 0) \cap ... \cap (g_L(\mathbf{X}) > 0)] \end{aligned} \tag{3-12}$$

Eq. (3-12) means that the reliability of a project duration is the event that all the L performance functions have to be larger than zero, that is the target duration t is larger than all the path durations $T_i(\mathbf{X})$, and it is equal to the event that the target duration t is larger than the maximum of $T_i(\mathbf{X})$. This means

$$P_S = Prob\{t - \max[T_1(\mathbf{X}), T_2(\mathbf{X}), ..., T_L(\mathbf{X})]] > 0\} \tag{3-13}$$

The corresponding failure probability of the project duration can be expressed as

$$P_F = Prob\{t - \max[T_1(\mathbf{X}), T_2(\mathbf{X}), ..., T_L(\mathbf{X})] \leq 0\} \tag{3-14}$$

Thus, the overall state performance function of the project duration, G, can be expressed as

$$G(\mathbf{X}) = t - \max[T_1(\mathbf{X}), T_2(\mathbf{X}), ..., T_L(\mathbf{X})] \tag{3-15}$$

3.3.2 Assessment of the first few moments of the project duration performance function

The computation of the reliability of project duration implicates the evaluation of the first three moments of the project duration performance function. Using the point-estimate method in independent standard normal space, the first three or four moments of the project duration performance function $G(\mathbf{X})$ can be estimated as follows [34-35]

$$\mu_G = \sum \prod_{i=1}^{n} P_{ci} \left\{ G\left[T^{-1}\left(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn}, t\right) \right] \right\} \tag{3-16}$$

$$\sigma_G^2 = \sum \prod_{i=1}^{n} P_{ci} \left\{ G\left[T^{-1}\left(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn}, t\right) - \mu_G \right] \right\}^2 \tag{3-17}$$

$$\alpha_{3G}\sigma_G^3 = \sum \prod_{i=1}^{n} P_{ci} \left\{ G\left[T^{-1}\left(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn}, t\right) - \mu_G \right] \right\}^3 \tag{3-18}$$

$$\alpha_{4G}\sigma_G^4 = \sum \prod_{i=1}^{n} P_{ci} \left\{ G\left[T^{-1}\left(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn}, t\right) - \mu_G \right] \right\}^4 \tag{3-19}$$

where n denotes the dimension of random vector $\mathbf{X}$; c indicates the distinct combination of n items from the group [1, 2, …, m]; m denotes the number of estimating points; ci denotes the ith term of c; u_{ci} and P_{ci} indicate the cith estimating points and the corresponding weights, respectively; μ_G, σ_G, α_{3G}, and α_{4G} denote the first four moments, i.e., mean, standard deviation, skewness, and kurtosis of $G(\mathbf{X})$, respectively, and T^{-1} denotes the Rosenblatt transformation.

Utilizing Eqs. (3-16)–(3-19), m^n function evaluations are necessary to determine the project completion time performance function $G(\mathbf{X})$; at large values of n, the calculation becomes heavy. Thus, the dimension reduction integration [36] is adopted to simplify the calculation. Considering the first four moments of $G(\mathbf{X})$, the bivariate-dimension reduction method [36] is introduced. The performance function $G(\mathbf{X})$ can then be approximated by $G^*(\mathbf{X})$ as

$$G(\mathbf{X}) \cong G^*(\mathbf{X}) = G^*[T^{-1}(\mathbf{U})] = \sum_{i<j} G_{i,j} - (n-2)\sum_{i=1}^{n} G_i + \frac{(n-1)(n-2)}{2} G_0 \tag{3-20}$$

where

$$G_{i,j} = G_{i,j}\left[\mu_1, \ldots, T^{-1}(u_i), \ldots, T^{-1}(u_j), \ldots, \mu_n, t\right] \tag{3-21}$$

$$G_i = G_i\left[\mu_1, \ldots, T^{-1}(u_i), \ldots, \mu_n, t\right] \tag{3-22}$$

$$G_0 = G\left(\mu_1, \ldots, \mu_i, \ldots, \mu_n, t\right) \tag{3-23}$$

where $G_{i,j}$ denotes a two-dimensional function in terms of $T^{-1}(u_i)$ and $T^{-1}(u_j)$, $i, j = 1, \ldots, n$, and $i < j$; G_i denotes a one-dimensional function of $T^{-1}(u_i)$, and G_0 denotes a constant; μ_i ($i = 1, \ldots, n$)

denote the mean value of random variables.

Note that $G^*(\mathbf{X})$ is represents a reduced integration, only one- and two-dimensional integrations are required, as opposed to one N-dimensional integration in $G(\mathbf{X})$, there is no need to calculate partial derivatives [36]. Therefore, the kth raw moments of $G(\mathbf{X})$ can be approximately formulated as

$$\begin{aligned}\mu_{kG} &= E\left\{\left[G(\mathbf{X})\right]^k\right\} \cong E\left\{\left[G^*(\mathbf{X})\right]^k\right\} = E\left\{\left\{G^*[T^{-1}(\mathbf{U})]\right\}^k\right\} \\ &\cong \sum_{i<j} \mu_{G_{i,j}}^k - (n-2)\sum_{i=1}^{n} \mu_{G_i}^k + \frac{(n-1)(n-2)}{2} G_0^k\end{aligned} \tag{3-24}$$

where

$$G_0^k = \left[G(\mu_1, ..., \mu_i, ..., \mu_n, t)\right]^k \tag{3-25}$$

$$\mu_{G_i}^k = \int_{-\infty}^{\infty} \left\{G_i\left[\mu_1, ..., T^{-1}(u_i), ..., \mu_n, t\right]\right\}^k \phi(u_i)\, du_i \tag{3-26}$$

$$\mu_{G_{i,j}}^k = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{G_{i,j}\left[\mu_1, ..., T^{-1}(u_i), ..., T^{-1}(u_j), ..., \mu_n\right]\right\}^k \phi(u_i)\phi(u_j)\, du_i du_j \tag{3-27}$$

and μ_{kG} ($k = 1, 2, 3, 4$) are the first four raw moments of the performance function $G(\mathbf{X})$.

Utilizing the point-estimate method [35], the one-dimensional integral in Eq. (3-26) can be approximated as

$$\mu_{G_i}^k = \sum_{r=1}^{m} P_r \left\{G_i\left[\mu_1, ..., T^{-1}(u_{ir}), ..., \mu_n, t\right]\right\}^k \tag{3-28}$$

where m denotes the number of estimated points; $u_{ir} = \sqrt{2}\,x_r$ and $P_r = w_r / \sqrt{\pi}$ denote the estimated points and corresponding weights, respectively, in which x_r and w_r are the abscissas and weights of the Gauss–Hermite quadrature with weight function $\exp(-x^2)$ [37].

Similarly, the two-dimensional integral in Eq. (3-27) can be estimated as

$$\mu_{G_{i,j}}^k = \sum_{r_1=1}^{m}\sum_{r_2=1}^{m} P_{r_1}P_{r_2} \left\{G_{i,j}\left[(\mu_1, ..., T^{-1}(u_{ir_1}), ..., T^{-1}(u_{jr_2}), ..., \mu_n, t)\right]\right\}^k \tag{3-29}$$

Based on the seven-point estimation ($m = 7$) of the standard normal distribution space, indicated by Zhao and Ono [35], the values of u_{iq} ($q = r, r_1, r_2$) and weight P_r are obtained as follows

$$u_{i1} = -3.7504397,\ P_1 = 5.48269\times10^{-4} \tag{3-30}$$

$$u_{i2} = -2.3667594,\ P_2 = 3.07571\times10^{-2} \tag{3-31}$$

$$u_{i3} = -1.1544054,\ P_3 = 0.2401233 \tag{3-32}$$

$$u_{i4} = 0,\ P_4 = 0.4571427 \tag{3-33}$$

$$u_{i5} = 1.1544054,\ P_5 = 0.2401233 \tag{3-34}$$

$$u_{i6} = 2.3667594,\ P_6 = 3.07571\times10^{-2} \tag{3-35}$$

$$u_{i7} = 3.7504397,\ P_7 = 5.48269\times10^{-4} \tag{3-36}$$

Finally, the first four moments (μ_G, σ_G, α_{3G}, and α_{4G}) of the performance function, described in Eq. (3-15), can be estimated as follows

$$\mu_G = \mu_{1G} \tag{3-37}$$

$$\sigma_G = \sqrt{\mu_{2G} - \mu_{1G}^2} \tag{3-38}$$

$$\alpha_{3G} = \left(\mu_{3G} - 3\mu_{2G}\mu_{1G} + 2\mu_{1G}^3\right)/\sigma_G^3 \tag{3-39}$$

$$\alpha_{4G} = \left(\mu_{4G} - 4\mu_{3G}\mu_{1G} + 6\mu_{2G}\mu_{1G}^2 - 3\mu_{1G}^4\right)/\sigma_G^4 \tag{3-40}$$

3.3.3 Estimating the reliability of the project duration

3.3.3.1 Estimating the reliability of the project duration using third-moment reliability index

After the first three central moments are obtained, the reliability of the project duration is estimated using the formulas of third-moment reliability index (β_{3M}) proposed by Zhao and Ono [38]

$$\beta_{3M} = \frac{-\text{sign}(\alpha_{3G})}{\sqrt{\text{In}(A)}}\text{In}[\sqrt{A}(1+\frac{\beta_{2M}}{u_b})] \tag{3-42}$$

where $\beta_{2M} = \mu_G / \sigma_G$ denotes the second-reliability index, and the parameters u_b and A can be obtained by

$$A = 1 + \frac{1}{u_b^2} \tag{3-43}$$

$$u_b = \sqrt[3]{a+b} + \sqrt[3]{a-b} - \frac{1}{\alpha_{3G}} \tag{3-44}$$

$$a = \frac{1}{\alpha_{3G}}(\frac{1}{2} + \frac{1}{\alpha_{3G}^2}), \qquad b = -\frac{1}{2\alpha_{3G}^2}\sqrt{\alpha_{3G}^2 + 4} \tag{3-45}$$

When $-1 \le \alpha_{3G} \le 1$, the third-moment reliability index in Eq. (3-40) can be simplified as [34]

$$\beta_{3M} = -\frac{\alpha_{3G}}{6} - \frac{3}{\alpha_{3G}}\ln(1 - \frac{1}{3}\alpha_{3G}\beta_{2M}) \tag{3-46}$$

The reliability of the project duration can be given as

$$P_S = 1 - P_F = 1 - \Phi(-\beta_{3M}) \tag{3-47}$$

3.3.3.2 Estimating the reliability of the project duration using fourth-moment normal

transformation

After the first four central moments are obtained, the reliability of the project completion time can be estimated using the fourth-moment normal transformation.

For a random variable G, i.e., if its first four moments μ_G, σ_G, α_{3G}, and α_{4G} are known, the standardized random variable G_s can be expressed by the fourth-moment normal transformation as follows [39]:

$$G_s = (G-\mu_G)\ /\ \sigma_G = S_u(u) = a_1 + a_2 u + a_3 u^2 + a_4 u^3 \tag{3-48}$$

where G_s is a standardized random variable, with the skewness and kurtosis being the same as those of G, $S_u(u)$ is a third-order polynomial of u, a_1, a_2, a_3, and a_4 are coefficients calculated by setting the first four moments of the left side of Eq. (3-48) equal to those of the right side. A detailed solution process is available from the study of Zhao et. al. [40], which is listed in the Appendix. Therefore, the inverse function of the relationship between the standard normal variable u and the standardized variable G_s can be expressed as

$$u = S^{-1}(G_s) \tag{3-49}$$

where S^{-1} denotes the inverse function of S, and the explicit expression of u are summarized in Table 3-1 according to Zhao et al. [40].

Table 3-1 Complete monotonic expressions of u related to G_s

Parameters			Range of G	Expression of u	Type
$a_4 < 0$			$J_2^* < G < J_1^*$	$-2r\cos[(\theta+\pi)/3]-a/3$	I
$a_4 > 0$	$P < 0$	$\alpha_{3G} \geq 0$	$J_1^* < G < J_2^*$	$2r\cos(\theta/3)-a/3$	II
			$G \geq J_2^*$	$\sqrt[3]{A}+\sqrt[3]{B}-a/3$	
		$\alpha_{3G} < 0$	$J_1^* < G < J_2^*$	$-2r\cos[(\theta-\pi)/3]-a/3$	III
			$G \leq J_1^*$	$\sqrt[3]{A}+\sqrt[3]{B}-a/3$	
	$P \geq 0$		$(-\infty,\infty)$	$\sqrt[3]{A}+\sqrt[3]{B}-a/3$	VI
$a_4 = 0$		$\alpha_{3G} \neq 0$	$a_2^2+4a_3(a_3+G_s)\geq 0$	$[-a_2+\sqrt{a_2^2+4a_3(a_3+G_s)}]/2a_3$	V
		$\alpha_{3G} = 0$	$(-\infty,\infty)$	G_s	IV

According to Table 3-1, the G-u transformation has six types, i.e., Type I, II, III, IV, V and VI, and each type has an applicable range of G. The values of u changing with G for the six types, respectively. By eliminating the unsuitable values of u, the change of the specific values of u with G, i.e., the G-u transformation, for each type is monotonic [40].

Subsequently, using Eqs. (3-50)–(3-55), the parameters p, r, θ, a, A, B, J_1^* and J_2^* of Table 3-1 can be obtained.

$$p = \frac{3a_2a_4 - a_3^2}{3a_4^2} \tag{3-50}$$

$$\Delta = (\frac{p}{3})^3 + (\frac{q}{2})^2,\ q = \frac{2a^3}{27} - \frac{ac}{3} - a - \frac{G_s}{a_4},\ a = \frac{a_3}{a_4},\ c = \frac{a_2}{a_4} \tag{3-51}$$

$$\theta = \arccos(\frac{-q}{2r^3}),\ r = \sqrt{-\frac{p}{3}} \tag{3-52}$$

$$\mathrm{A} = -\frac{q}{2} + \sqrt{\Delta},\ \mathrm{B} = -\frac{q}{2} - \sqrt{\Delta} \tag{3-53}$$

$$J_1^* = \sigma_G a_4(-2r^3 + \frac{2a^3}{27} - \frac{ac}{3} - a) + \mu_G \tag{3-54}$$

$$J_2^* = \sigma_G a_4(2r^3 + \frac{2a^3}{27} - \frac{ac}{3} - a) + \mu_G \tag{3-55}$$

According to Eq. (3-48) the CDF and PDF are expressed as [41]

$$F_{G_s}(G_s) = \Phi(u) \tag{3-56}$$

$$f(G_s) = \frac{\phi(u)}{\sigma_x(3a_4u^2 + 2a_3u + a_2)} \tag{3-57}$$

Substituting Eq. (3-48) with an explicit expression of u listed in Table 3-1 into Eq. (3-56) yields

$$F_{G_s}(G_s) = \Phi(u) = \Phi[S^{-1}(G_s)] \tag{3-58}$$

Therefore, the reliability of G can be expressed as

$$P_S = 1 - P_F = 1 - P[G \le 0] = 1 - F_{G_s}(\frac{-\mu_G}{\sigma_G}) = 1 - \Phi[S^{-1}(\frac{-\mu_G}{\sigma_G})] \tag{3-59}$$

3.4 Verification of the proposed method

In this section, three examples are considered to illustrate the efficiency and effectiveness of the proposed methods. First, a simple project network with three paths is considered, in which the performance function and its first three moments are expressed explicitly to present the algorithmic procedure in detail. In the second example, the proposed method is applied to an actual large-scale road pavement project involving nine paths. Then the reliability of the project completion time is estimated based on the third-moment reliability index. In the third example, an industrial building project involving the construction of a single storey industrial building with an adjoining parking lot is analyzed. And the reliability of the project completion time is estimated by the fourth-moment normal transformation. Recent studies have shown that lognormal distribution is more appropriate for estimating the duration distribution of each activity [42-45], so that it is adopted in these two examples.

3.4.1 Example 1: a simple project network

In the first example, a simple project with three paths, as presented in Figure 3-1, is considered. The mean and variance associated with the duration of each activity is presented in Table 3-2, and three activity network paths are presented in Table 3-3. The target duration of this project is 70 d (d = in days) in this example.

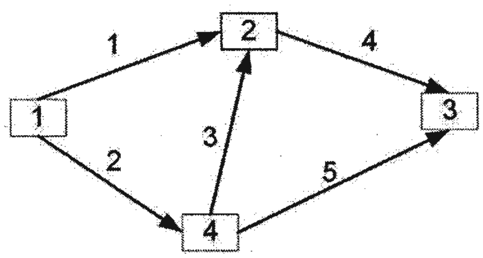

Figure 3-1. A considered simple project network

Table 3-2. Statistics of each activity duration (d)

Activity	Mean (μ)	Deviation (s^2)	Distribution
1	32	36.0	Lognormal
2	20	13.0	Lognormal
3	16	4.00	Lognormal
4	18	6.25	Lognormal
5	36	16.0	Lognormal

Table 3-3. Activity network paths for the simple project

Path	Activities of each path
1	1, 4
2	2, 3, 4
3	2, 5

According to Eq. (3-15), the overall performance function with respect to the project duration can be expressed as

$$G(\mathbf{X}) = 70 - \max[T_1(\mathbf{X}), T_2(\mathbf{X}), T_3(\mathbf{X})] \tag{3-60}$$

Then, using bivariate-dimension reduction method, as shown in Eq. (3-20), the overall performance function $G(\mathbf{X})$ in Eq. (3-60) can be approximated as

$$G(\mathbf{X}) \cong G^{*}(\mathbf{X}) = G^{*}[T^{-1}(\mathbf{U})] = \sum_{i<j} G_{i,j} - 3\sum_{i=1}^{5} G_i + 6G_0 \tag{3-61}$$

where

$$G_{1,2} = 70 - \max[X_1 + 18, X_2 + 36],\quad G_{1,3} = 70 - \max[X_1 + 18, X_3 + 38, 56],$$

$$G_{1,4} = 70 - \max[X_1 + X_4, X_4 + 36, 56],\quad G_{1,5} = 70 - \max[X_1 + 18, 54, X_5 + 20],$$

$$G_{2,3} = 70 - \max[50, X_2 + X_3 + 18, X_2 + 36],\quad G_{2,4} = 70 - \max[X_4 + 32, X_2 + X_4 + 16, X_2 + 36],$$

$$G_{2,5} = 70 - \max[50, X_2 + 34, X_2 + X_5],\quad G_{3,4} = 70 - \max[32 + X_4, 20 + X_3 + X_4, 56],$$

$$G_{3,5} = 70 - \max[50, X_3 + 38, 20 + X_5],\quad G_{4,5} = 70 - \max[32 + X_4, X_3 + 36, 20 + X_5],$$

$$G_1 = 70 - \max[X_1 + 18, 56],\quad G_2 = 70 - \max[X_2 + 36, 50],$$

$$G_3 = 70 - \max[X_3 + 38, 56],\quad G_4 = 70 - \max[X_4 + 36, 56],$$

$$G_5 = 70 - \max[X_5 + 20, 54],\quad G_0 = 14.$$

Based on the statistical information of the random variables shown in Table 3-2, by using the seven-point estimate given in Eqs. (3-30)-(3-36), the corresponding original space estimating points of these random variables can be obtained, as shown in Table 3.

Utilizing the original space estimating points indicated in Table 3-4 and Eq. (3-28), $\mu_{G_1}^{1}$ can be obtained as

$$\mu_{G_1}^{1} = \sum_{r=1}^{7} P_r \left\{ G_1\left[T^{-1}(u_{1r}), \mu_2, \mu_3, \mu_4, \mu_5, 70\right] \right\} = 13.418 \tag{3-62}$$

Table 3-4. Original space values of the random variable X_i (d)

X_i	x_{i1}	x_{i2}	x_{i3}	x_{i4}	x_{i5}	x_{i6}	x_{i7}
X_1	15.663	20.257	25.378	31.452	38.980	48.833	63.156
X_2	1.810	4.114	8.450	16.769	33.276	68.343	155.393
X_3	6.165	8.667	11.682	15.522	20.625	27.800	39.084
X_4	4.798	7.652	11.519	17.004	25.102	37.787	60.269
X_5	6.693	12.044	20.151	32.897	53.706	89.860	161.697

Similarly, $\mu_{G_2}^{1}$, $\mu_{G_3}^{1}$, $\mu_{G_4}^{1}$, and $\mu_{G_5}^{1}$ can also be obtained, which are listed in Table 3-5.

Table 3-5. Results of $\mu^1_{G_i}(d)$

Values	Results of $\mu^1_{G_i}$
$i=1$	13.418
$i=2$	12.357
$i=3$	13.057
$i=4$	12.206
$i=5$	9.480

Then, substituting the original space estimating points in Table 3-4 into Eq. (3-29), the value of $\mu^1_{G_{1,2}}$ can be calculated as

$$\begin{aligned}\mu^1_{G_{1,2}} &= \sum_{r_1=1}^{7}\sum_{r_2=1}^{7} P_{r_1}P_{r_2}\left\{G_{1,2}\left[(T^{-1}(u_{1r_1}),T^{-1}(u_{2r_2}),\mu_3,\mu_4,\mu_5,70)\right]\right\} \\ &= \sum_{r_1=1}^{7}\sum_{r_2=1}^{7} P_{r_1}P_{r_2}\left\{70-\max[T^{-1}(u_{1r_1})+18,T^{-1}(u_{2r_2})+36]\right\} = 11.577\end{aligned} \tag{3-63}$$

Similarly, $\mu^1_{G_{i,j}}$ $(i<j)$, $(i,j=1, \ldots, 5)$ can be obtained, which are shown in Table 3-6.

Furthermore, G_0^2, G_0^3, $\mu^2_{G_i}$, $\mu^3_{G_i}$, $\mu^2_{G_{i,j}}(i<j)$ and $\mu^3_{G_{i,j}}(i<j)$ can also be computed in the same way. Then, combining these values with Eqs. (3-24)–(3-29), the first three raw moments of $G(\mathbf{X})$ in Eq. (3-47) can be obtained

$$\mu_{1G} = \sum_{i<j}\mu^1_{G_{i,j}} - 3\sum_{i=1}^{5}\mu^1_{G_i} + 6G_0 = 7.019 \tag{3-64}$$

$$\mu_{2G} = \sum_{i<j}\mu^2_{G_{i,j}} - 3\sum_{i=1}^{5}\mu^2_{G_i} + 6G_0^2 = 369.859 \tag{3-65}$$

$$\mu_{3G} = \sum_{i<j}\mu^3_{G_{i,j}} - 3\sum_{i=1}^{5}\mu^3_{G_i} + 6G_0^3 = -2128.75 \tag{3-66}$$

Table 3-6. Results of $\mu^k_{G_{i,j}}$ (d)

Values	$i=1$	$i=2$	$i=3$	$i=4$
$j=2$	11.577			
$j=3$	12.568	11.636		
$j=4$	11.721	10.959	11.822	
$j=5$	8.659	7.887	8.988	8.753

Substituting

the values of μ_{1G}, μ_{2G}, and μ_{3G} into Eqs. (3-37)–(3-39), the first three moments (μ_G, σ_G, α_{3G}) can be obtained as

$$\mu_G = \mu_{1G} = 7.019 \tag{3-67}$$

$$\sigma_G = \sqrt{\mu_{2G} - \mu_{1G}^2} = 17.905 \tag{3-68}$$

$$\alpha_{3G} = \left(\mu_{3G} - 3\mu_{2G}\mu_{1G} + 2\mu_{1G}^3\right) / \sigma_G^3 = -1.607 \tag{3-69}$$

In this case, because $\alpha_{3G} < -1$, the third-moment reliability index of the project duration should be calculated according to Eq. (3-41), which is given as

$$\beta_{3M} = \frac{-\text{sign}(\alpha_{3G})}{\sqrt{\ln(A)}} \ln[\sqrt{A}(1 + \frac{\beta_{2M}}{u_b})] = 0.613 \tag{3-70}$$

Based on Eq. (3-47), the reliability of the project duration can be estimated as

$$P_S = 1 - P_F = 1 - \Phi(-\beta_{3M}) = 0.730 \tag{3-71}$$

Table 3-7. Failure probability and reliability of the project duration (d)

	μ_G	σ_G	α_{3G}	P_S
Proposed method	5-18	17.905	-1.607	0.730
MCS				0.734

Table 3-7 lists the failure probability and the reliability of the project duration obtained by using the MCS with 1,000,000 samples and the proposed method.

One can see from Table 3-7 that the results obtained from the proposed method are nearly identical to those obtained from using MCS. However, the proposed method can generate results with only 1429 times, resulting in significantly less computation than that associated with MCS. In addition, FARB is not applicable for such small project because the number of activities on each path is less than five.

3.4.2 Example 2: a large-scale road pavement project network

In the second example, an actual large-scale road pavement project is considered, which was first investigated by Brook et al. [46]. Figure 3-3 indicates the network diagram adopted in the project.

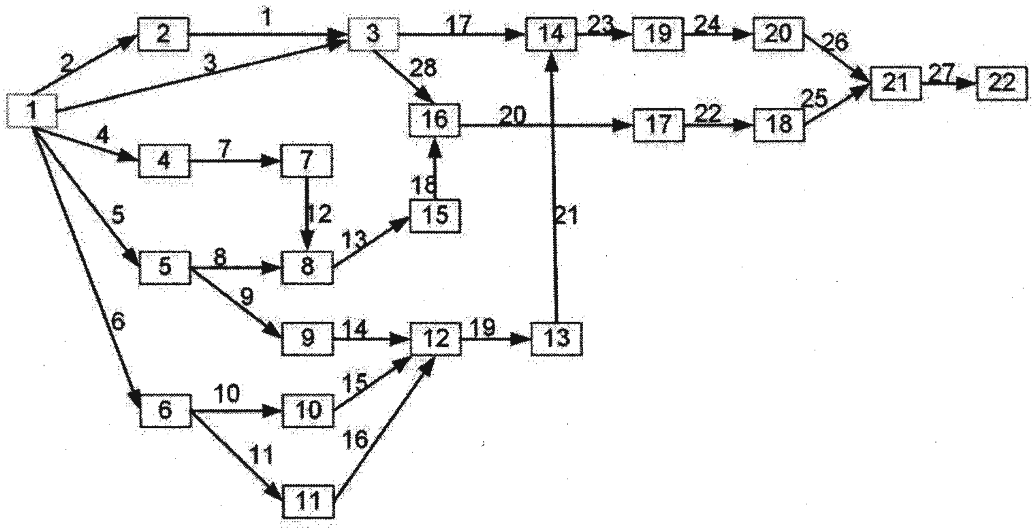

Figure 3-3. Activity network of the large-scale road pavement project

The activities performed in this project, include paving 2.2 miles of road, constructing appurtenant drainage structures, excavating to grade, placing macadam shoulders, erecting guardrails, and landscaping; more detailed information has been presented by Guo et al. [23]. The mean and standard deviation with respect to the duration of all activities in days are presented in Table 3-8, and Table 3-9 presents all the paths and activities comprising the network plans.

Table 3-8. Statistical data of the duration of the activities (*d*)

Activity	μ	σ	Distribution	Activity	μ	σ	Distribution
1	0	0.0	Lognormal	15	9.0	4.50	Lognormal
2	2	0.5	Lognormal	16	6.0	2.00	Lognormal
3	5	1.0	Lognormal	17	2.0	0.50	Lognormal
4	6	1.5	Lognormal	18	7.0	1.73	Lognormal
5	3	0.5	Lognormal	19	5.0	2.00	Lognormal
6	7	4.0	Lognormal	20	10.0	2.00	Lognormal
7	10	2.0	Lognormal	21	7.0	3.31	Lognormal
8	3	1.0	Lognormal	22	6.0	1.50	Lognormal
9	7	1.5	Lognormal	23	10.0	4.50	Lognormal
10	5	2.0	Lognormal	24	6.0	1.50	Lognormal
11	3	1.5	Lognormal	25	3.0	1.00	Lognormal
12	9	2.0	Lognormal	26	3.0	1.00	Lognormal
13	5	1.5	Lognormal	27	5.0	1.50	Lognormal
14	3	0.5	Lognormal	28	0.0	0.00	Lognormal

Table 3-9. Activity network paths of the case project

Path	Activities of each path
1	2, 17, 1, 23, 24, 26, 27
2	2, 17, 28, 20, 22, 25, 27
3	3, 1, 23, 24, 26, 27
4	3, 28, 20, 22, 25, 27
5	4, 7, 12, 13, 18, 20, 22, 25, 27
6	5, 8, 13, 18, 20, 22, 25, 27
7	5, 9, 14, 19, 21, 23, 24, 26, 27
8	6, 10, 15, 19, 21, 23, 24, 26, 27
9	6, 11, 16, 19, 21, 23, 24, 26, 27

The overall performance function $G(\mathbf{X})$ with respect to the project duration can be defined according to Eq. (3-15) as

$$G(\mathbf{X}) = t - \max[T_1(\mathbf{X}), T_2(\mathbf{X}), T_3(\mathbf{X}), T_4(\mathbf{X}), T_5(\mathbf{X}), T_6(\mathbf{X}), T_7(\mathbf{X}), T_8(\mathbf{X}), T_9(\mathbf{X})] \tag{3-72}$$

where

$$T_1(\mathbf{X}) = X_2 + X_{17} + X_1 + X_{23} + X_{24} + X_{26} + X_{27}$$

$$T_2(\mathbf{X}) = X_2 + X_{17} + X_{28} + X_{20} + X_{22} + X_{25} + X_{27}$$

$$T_3(\mathbf{X}) = X_3 + X_1 + X_{23} + X_{24} + X_{26} + X_{27}$$

$$T_4(\mathbf{X}) = X_3 + X_{28} + X_{20} + X_{22} + X_{25} + X_{27}$$

$$T_5(\mathbf{X}) = X_4 + X_7 + X_{12} + X_{13} + X_{18} + X_{20} + X_{22} + X_{25} + X_{27}$$

$$T_6(\mathbf{X}) = X_5 + X_8 + X_{13} + X_{18} + X_{20} + X_{22} + X_{25} + X_{27}$$

$$T_7(\mathbf{X}) = X_5 + X_9 + X_{14} + X_{19} + X_{21} + X_{23} + X_{24} + X_{26} + X_{27}$$

$$T_8(\mathbf{X}) = X_6 + X_{10} + X_{15} + X_{19} + X_{21} + X_{23} + X_{24} + X_{26} + X_{27}$$

$$T_9(\mathbf{X}) = X_6 + X_{11} + X_{16} + X_{19} + X_{21} + X_{23} + X_{24} + X_{26} + X_{27}$$

According to Eq. (3-20), the overall performance function $G(\mathbf{X})$ in Eq. (3-72) can be approximated as

$$G(\mathbf{X}) \cong G^*(\mathbf{X}) = G^*[T^{-1}(\mathbf{U})] = \sum_{i<j} G_{i,j} - 24\sum_{i=1}^{26} G_i + 300G_0 \tag{3-73}$$

For different target durations t, substituting the mean of all random variables in Table 7 into Eq.

(3-25), G_0^k $(k = 1, 2, 3)$ can be obtained

$$\begin{aligned} G_0^k &= \left[G_\mu(\mu_1,\dots,\mu_i,\dots,\mu_{26},t)\right]^k \\ &= \{t - \max[T_1(\boldsymbol{\mu}),T_2(\boldsymbol{\mu}),T_3(\boldsymbol{\mu}),T_4(\boldsymbol{\mu}),T_5(\boldsymbol{\mu}),T_6(\boldsymbol{\mu}),T_7(\boldsymbol{\mu}),T_8(\boldsymbol{\mu}),T_9(\boldsymbol{\mu})]\}^k \\ &= (t-61)^k \end{aligned} \tag{3-74}$$

Using the eleven-point estimate in standard normal space and the statistical information of each random variable in Table 3-8, the corresponding original space estimating points can be obtained via the inverse Rosenblatt transformation. Substituting all original space estimating points into Eq. (3-28), the values of $\mu_{G_i}^k$ $(k = 1, 2, 3)$ are obtained.

Similarly, the original spatial values associated with any pair of random variables $T^{-1}(u_{ir_1})$, and $T^{-1}(u_{jr_2})$ are inserted into Eq. (3-29), $\mu_{G_{i,j}}^k \left(i<j\right)$ $(k = 1, 2, 3)$ are computed.

By substituting $\mu_{G_0}^k, \mu_{G_i}^k, \mu_{G_{i,j}}^k$ $(k = 1, 2, 3)$ into Eqs. (3-24)–(3-29), the values of μ_{kG} $(k = 1, 2, 3)$ are obtained as follows

$$\mu_{1G} = \sum_{i<j} \mu_{G_{i,j}}^1 - 24\sum_{i=1}^{26} \mu_{G_i}^1 + 300G_0 = t - 63.695 \tag{3-75}$$

$$\mu_{2G} = \sum_{i<j} \mu_{G_{i,j}}^2 - 24\sum_{i=1}^{26} \mu_{G_i}^2 + 300G_0^2 = 4090.27 + t(t-127.39) \tag{3-76}$$

$$\mu_{3G} = \sum_{i<j} \mu_{G_{i,j}}^3 - 24\sum_{i=1}^{26} \mu_{G_i}^3 + 300G_0^3 = 12270.8t + t^2(t-191.084) - 264.873 \tag{3-77}$$

Based on the first three raw moments and Eqs. (3-37)–(39), the first three moments (μ_G, σ_G, α_{3G}) of the performance function under different target duration t can be obtained

$$\mu_G = \mu_{1G} = t - 63.695 \tag{3-78}$$

$$\sigma_G = \sqrt{\left(\mu_{2G}\right) - \mu_{1G}^2} = 5.766 \tag{3-79}$$

$$\alpha_{3G} = \left(\mu_{3G} - 3\mu_{2G}\mu_{1G} + 2\mu_{1G}^3\right)/\sigma_G^3 = -0.569 \tag{3-80}$$

According to Eq. (3-45) the reliability index under different target duration t can be obtained

$$\beta_{3M} = -\frac{\alpha_{3G}}{6} - \frac{3}{\alpha_{3G}}\ln\left(1 - \frac{1}{3}\alpha_{3G}\beta_{2M}\right) = 0.0948 + 5.272\ln(0.0329t - 1.095) \tag{3-81}$$

The reliability of the project duration can be then obtained via Eq. (3-46)

$$P_S = 1 - P_F = 1 - \Phi(-\beta_{3M}) \quad (3\text{-}82)$$

With the explicit expression of Eqs. (81)–(82), the reliability curve of the project duration under different target duration t can be given, as presented in Figure 3-4. So that reliability of the large-scale road pavement project network can be quantitatively analyzed.

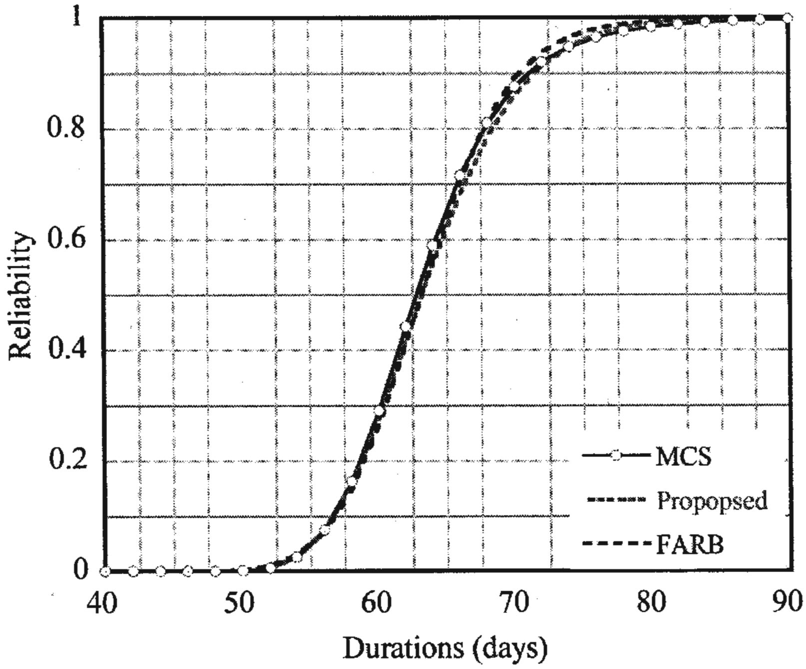

Figure 3-4. Reliability of the large-scale road pavement project network

In Figure 3-4, the results of the reliability of the project duration obtained using the proposed method are compared with those generated from the FARB method and MCS with 1,000,000 samples. One can see that the results obtained using the proposed method are consistent with those obtained using MCS and FARB method. For example, the calculated reliability using MCS, the proposed method, and FARB are 0.0461, 0.0464 and 0.0474, respectively, when considering a target duration of 55 days. However, unlike the FARB method, the proposed method does not require the calculations of the correlation coefficients between any pair of paths and the joint failure probability of any pair of representative paths.

3.4.3 Example 3: an industrial building project

In this section, an industrial building project involving the construction of a single storey industrial building with an adjoining parking lot is analyzed. This project, which was first investigated by Brend et al. [47], comprises reinforced concrete piers, frost walls, structural steel columns, and a precast roof deck. And it has 38 activities and 33 paths. There are 8 dummy activities, which describe the sequential logical relationship between activities. The means and standard deviations of the respective activity durations are listed in Table 3-10. More details regarding the various activities are available in the study of Guo et.al. [23]. The corresponding project network is shown in Fig.3-5 (where ○ is the number of code), and all the paths of this project network with the corresponding activities are listed in Table 3-11.

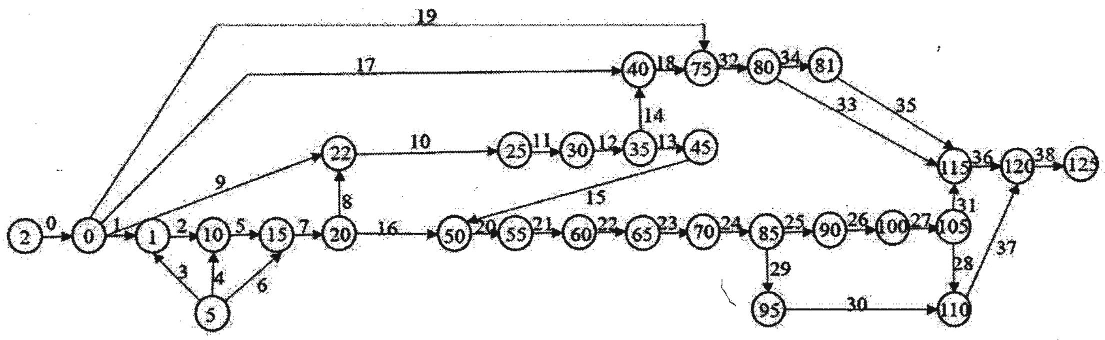

Fig.3-5 An industrial building project network

Table 3-10 Statistical data of activity durations (days)

Activity	μ	σ	Distribution
0	0	0	Lognormal
1	32	3.2	Lognormal
2	2	0.5	Lognormal
3	2	0.5	Lognormal
4	0	0	Lognormal
5	2	0.5	Lognormal
6	1	0.5	Lognormal
7	2	1	Lognormal
8	0	0	Lognormal
9	40	12	Lognormal
10	2	0.5	Lognormal
11	2	0.5	Lognormal
12	4	0.8	Lognormal
13	1	0.1	Lognormal
14	0	0	Lognormal
15	0	0	Lognormal
16	1	0.5	Lognormal
17	60	12	Lognormal

Table 3-10 Continued

Activity	μ	σ	Distribution
18	5	1	Lognormal
19	30	6	Lognormal
20	3	0.9	Lognormal
21	1	0.3	Lognormal
22	4	0.4	Lognormal
23	1	0.1	Lognormal
24	2	0.5	Lognormal
25	2	0.2	Lognormal
26	2	0.5	Lognormal
27	2	0.5	Lognormal
28	0	0	Lognormal
29	2	0.2	Lognormal
30	1	0.2	Lognormal
31	0	0	Lognormal
32	5	1.5	Lognormal
33	6	1.2	Lognormal
34	0	0	Lognormal
35	4	1.2	Lognormal
36	2	0.5	Lognormal
37	3	0.3	Lognormal
38	0	0	Lognormal

The overall performance function $G(\mathbf{X})$ of the project completion time can be defined according to Eq. (3-15) as

$$G(\mathbf{X}) = t - \max[T_1(\mathbf{X}),...,T_L(\mathbf{X})],\ (L = 1,...,33) \tag{3-83}$$

Based on Eq. (3-17), $G(\mathbf{X})$ in Eq. (3-83) can be approximated as

$$G(\mathbf{X}) \cong G^*(\mathbf{X}) = G^*[T^{-1}(\mathbf{U})] = \sum_{i<j} G_{i,j} - 28\sum_{i=1}^{30} G_i + 420G_0 \tag{3-84}$$

For different target durations t, by substituting all the means of the random variables in Table 2 into Eq. (3-24), G_0^k $(k = 1, 2, 3, 4)$ can be obtained.

$$\begin{aligned} G_0^k &= \left[G_\mu(\mu_1,...,\mu_i,...,\mu_{30},t)\right]^k \\ &= \{t - \max[T_1(\boldsymbol{\mu}),...,T_{33}(\boldsymbol{\mu})]\}^k = (t - 76.35)^k \end{aligned} \tag{3-85}$$

Using the seven-point estimate in standard normal space as expressed by Eqs. (3-30)–(3-36) and ie statistical information of each random variable in Table 3-10, the corresponding original space stimating points of random variables can be obtained via the inverse Rosenblatt transformation.

Table 3-11 Activity network paths of the industrial project

Path	Activities of each path
1	19,32,34,35,36
2	19,32,33,36
3	17,18,32,33,36
4	17,18,32,34,35,36
5	9,10,11,12,14,18,32,34,35,36
6	9,10,11,12,14,18,32,33,36
7	9,10,11,12,13,15,20,21,22,23,24,25,26,27,31,36
8	9,10,11,12,13,15,20,21,22,23,24,25,26,27,28,37
9	9,10,11,12,13,15,20,21,22,23,24,29,30,37
10	1,2,5,7,8,10,11,12,14,18,32,34,35,36
11	1,2,5,7,8,10,11,12,14,18,32,33,36
12	1,2,5,7,8,10,11,12,13,15,20,21,22,23,24,25,26,27,31,36
13	1,2,5,7,8,10,11,12,13,15,20,21,22,23,24,25,26,27,28,37
14	1,2,5,7,8,10,11,12,13,15,20,21,22,23,24,29,30,37
15	1,2,5,7,16,20,21,22,23,24,25,26,27,31,36
16	1,2,5,7,16,20,21,22,23,24,25,26,27,28,37
17	1,2,5,7,16,20,21,22,23,24,29,30,37
18	1,3,4,5,7,8,10,11,12,14,18,32,34,35,36
19	1,3,4,5,7,8,10,11,12,14,18,32,33,36
20	1,3,4,5,7,8,10,11,12,13,15,20,21,22,23,24,25,26,27,31,36
21	1,3,4,5,7,8,10,11,12,13,15,20,21,22,23,24,25,26,27,28,37
22	1,3,4,5,7,8,10,11,12,13,15,20,21,22,23,24,29,30,37
23	1,3,4,5,7,16,20,21,21,22,23,24,25,26,27,31,36
24	1,3,4,5,7,16,20,21,22,23,24,25,26,27,28,37
25	1,3,4,5,7,16,20,21,22,23,24,29,30,37
26	1,3,6,7,8,10,11,12,14,18,32,34,35,36
27	1,3,6,7,8,10,11,12,14,18,32,33,36
28	1,3,6,7,8,10,11,12,13,15,20,21,22,23,24,25,26,27,31,36

Table 3-11 Continued

Path	Activities of each path
29	1,3,6,7,8,10,11,12,13,15,20,21,22,23,24,25,26,27,28,37
30	1,3,6,7,8,10,11,12,13,15,20,21,22,23,24,29,30,37
31	1,3,6,7,16,20,21,21,22,23,24,25,26,27,31,36
32	1,3,6,7,16,20,21,22,23,24,25,26,27,28,37
33	1,3,6,7,16,20,21,22,23,24,29,30,37

Substituting all the original space estimating points of G_i (i = 1, …, 30) into Eq. (3-26), the values of $\mu_{G_i}^k$ (k = 1, 2, 3, 4) are obtained.

Subsequently, the original spatial values associated with any pair of random variables $T^1(u_{ir_1})$, and $T^1(u_{jr_2})$ are inseted into Eq. (3-27), and $\mu_{G_{i,j}}^k$ $(i<j)$ (k = 1, 2, 3, 4) are computed.

Combining $\mu_{G_0}^k, \mu_{G_i}^k, \mu_{G_{i,j}}^k$ (k = 1, 2, 3, 4) with Eqs. (3-22)–(3-28), the values of u_{kG} (k = 1, 2, 3, 4) are obtained.

Based on the obtained first four raw moments, the first four moments of the performance function can be obtained as follows

$$\mu_G = \mu_{1G} = t - 81.960 \tag{3-86}$$

$$\sigma_G = \sqrt{\mu_{2G} - \mu_{1G}^2} = 10.865 \tag{3-87}$$

$$\alpha_{3G} = \left(\mu_{3G} - 3\mu_{2G}\mu_{1G} + 2\mu_{1G}^3\right)/\sigma_G^3 = -0.995 \tag{3-88}$$

$$\alpha_{4G} = \left(\mu_{4G} - 4\mu_{3G}\mu_{1G} + 6\mu_{2G}\mu_{1G}^2 - 3\mu_{1G}^4\right)/\sigma_G^4 = 4.407 \tag{3-89}$$

Utilizing the first four moments of G, the coefficients defined in Eq. (3-48) can be obtained as a_1 = 0.166, a_2 = 0.962, a_3 = –0.166, and a_4 = 0.003.

Then based on Eq. (3-50), the value of P can be obtained

$$p = \frac{3a_2a_4 - a_3^2}{3a_4^2} = -562.869 \tag{3-90}$$

Because $a_4 > 0$, $P < 0$ and $\alpha_{3G} < 0$, the expression of u belongs to Type III of Table 1. With Eqs. (3-51)–(3-55), the parameters r, θ, a, A, B, J_1^* and J_2^* can be calculated

$$\Delta = 5.444\times10^6 - 97783t + 198.4t^2,\ q = -6942.148 + 28.171t \tag{3-91}$$

$$a = -50.718,\ c = 294.553 \tag{3-92}$$

$$\theta = 5.444\times10^6 - 97783t + 198.4t^2,\ r = 13.698 \tag{3-93}$$

$$\mathrm{A} = 3471.07 - 14.085t + \sqrt{5.444\times10^6 - 97783t + 198.4t^2} \tag{3-94}$$

$$B = 3471.07 - 14.085t - \sqrt{5.444\times10^{6} - 97783t + 198.4t^{2}} \quad (3\text{-}95)$$

$$J_1^* = -1960.82 + 17.906t \quad (3\text{-}96)$$

$$J_2^* = -1595.91 + 17.906t \quad (3\text{-}97)$$

Subsequently, according to Eq. (3-59), the reliability of the project completion time under different target durations t can be obtained, as presented in Fig. 3-6.

$$P_S = 1 - P_F = 1 - F_{G_s}\left(\frac{81.96-t}{10.865}\right) = 1 - \Phi\left[S^{-1}\left(\frac{81.96-t}{10.865}\right)\right] \quad (3\text{-}98)$$

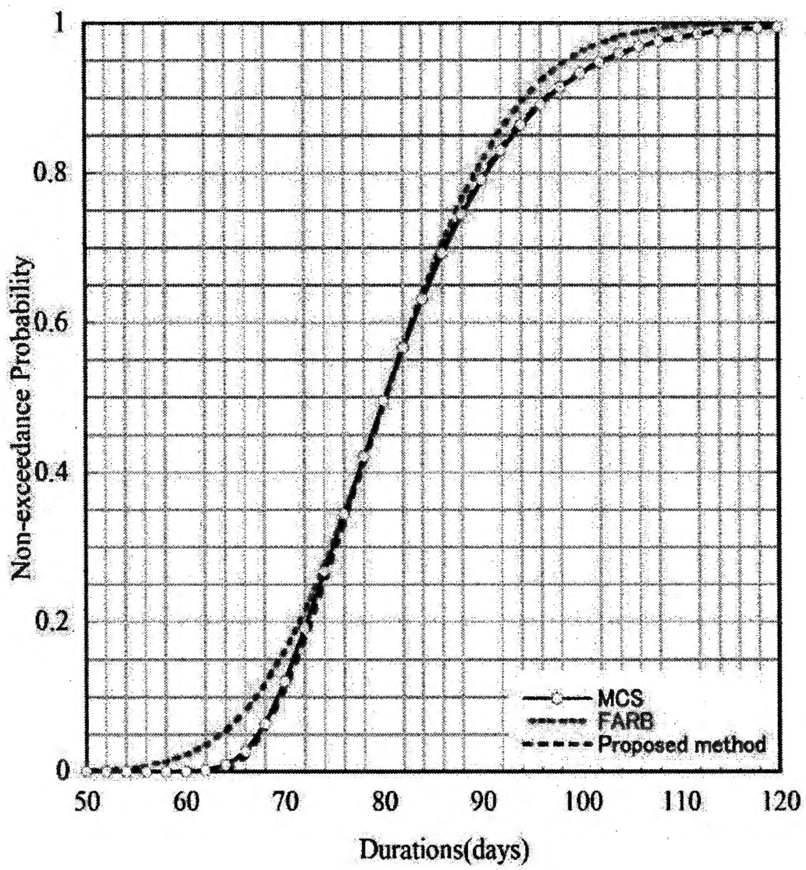

Fig. 3-6 Results comparison between different methods

Fig.3-6 presents the results of the reliability of the project completion time obtained using the proposed method, FARB method and the MCS with 1,000,000 samples, respectively.

As shown, the results obtained using the proposed method agree well with those obtained using MCS. However, the proposed method can generate results with only 1429 times, resulting in significantly less computation than that associated with MCS. Furthermore, unlike the FARB method, the proposed method does not require the calculations of the correlation coefficients between any pair of paths and the joint failure probability of any pair of representative paths.

3.5 Conclusions

In this study, simple and effective methods based on the method of moments have been proposed to assess the reliability of the project duration. The proposed methods consist of three main sections: first, an overall performance function is established with respect to the project duration; second, the bivariate-dimension reduction is used to evaluate the first three or four moments of this performance function; third, the reliability of the project duration is assessed by the third-moment reliability index or fourth-moment normal transformation.

Three numerical examples, including a simple project, a practical large-scale road pavement project and an industrial building project, were then used to demonstrate the efficiency and effectiveness of the proposed methods. It can be found that: the proposed methods can provide nearly the same reliability assessment results of the project duration compared with the MCS method with less calculation. And compared with the FARB method, the proposed methods do not require the calculations of the correlation coefficients between any pair of paths and the joint failure probability of any pair of representative paths. In addition, the proposed methods can give an explicit formula of project duration reliability curve under different target durations, avoiding repeated calculations as the target duration changes.

Appendix A Computation of the four coefficients of a_1, a_2, a_3, a_4

According to Fleishman [39], the four coefficients of a_1, a_2, a_3 and a_4 of Eq. (3-48) can be calculated by letting the first four moments of the left side of Eq. (3-48) equal to those of the right side

Mean:

$$\mu_G = 0 = E[S_u(u)] = a_1 + a_3 \tag{A-1}$$

Standard deviation:

$$\sigma_G = 1 = E[S_u^2(u)] = a_2^2 + 2a_3^2 + 6a_2a_4 + 15a_4^2 \tag{A-2}$$

Skewness:

$$\alpha_{3G} = E[S_u^3(u)] = 6a_2a_3 + 8a_3^3 + 72a_2a_3a_4 + 270a_3a_4^2 \tag{A-3}$$

Kurtosis:

$$\alpha_{4G} = E[S_u^4(u)] = 3(a_2^4 + 20a_2^3 a_4 + 210a_2^2 a_4^2 + 3465a_4^4) \\ + 12a_3^2(5a_2^2 + 5a_3^2 + 78a_2 a_4 + 375a_4^2) \tag{A-4}$$

By simplifying the equations above, parameters a_2 and a_4 can be obtained as follows

$$\alpha_{3G}^2 = 2A_1 A_2^2 \tag{A-5a}$$

$$\alpha_{4G} = 3A_1 A_3 + 3A_4 \tag{A-5b}$$

where

$$A_1 = 1 - a_2^2 - 6a_2 a_4 - 15a_4^2 \tag{A-5c}$$

$$A_2 = 2 + a_2^2 + 24a_2 a_4 + 105a_4^2 \tag{A-5d}$$

$$A_3 = 5 + 5a_2^2 + 126a_2 a_4 + 675a_4^2 \tag{A-5e}$$

$$A_4 = a_2^4 + 20a_2^3 a_4 + 210a_2^2 a_4^2 + 1260a_2 a_4^3 \tag{A-5f}$$

Due to the values of α_{3G}, and α_{4G} are known, the parameters a_2 and a_4 can be obtained from Eqs. (A-5a)–(A-5f), which can be solved by an appropriate nonlinear equation solver, with preconditions:

(I) All of the parameters are real numbers;

(II) A_1 is no less than zero ($A_1 = 2\alpha_{3G}^2$);

(III) When $\alpha_{3G} = 0$, $\alpha_{4G} = 3$, $a_3 = a_4 = 0$ and $a_2 = 1$. (To make sure that the fourth-moment transformation includes normal distribution.)

After the parameters a_2 and a_4 have been determined, the parameters a_1 and a_3 can be readily obtained

$$a_3 = -a_1 = \frac{\alpha_{3G}}{2A_2} \tag{A-5g}$$

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CHAPTER 4

Reliability analysis of project duration under the influence of single risk factor

4.1 Introduction

4.1.1 Research background and objective

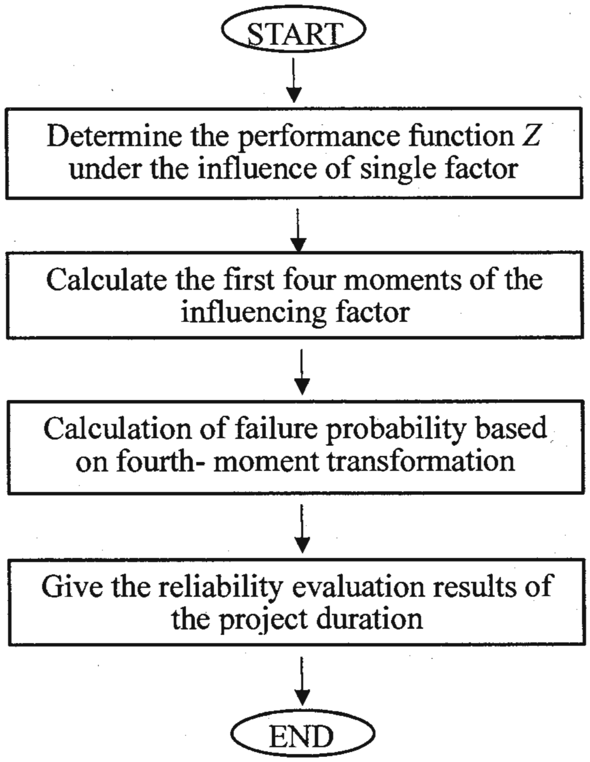
In recent years, Building Information Modeling (BIM) is wildly used in project duration management. In order to assess the reasonable level of the BIM-based project duration plan, some researchers have introduced reliability theory to evaluate. Although they combined project duration management with BIM from different perspectives, they did not take the impact of random factors on project duration into account [1-5]. Li [6] proposed a random prediction model for project duration and carried on the integrated management on the project duration management platform based on the BIM technology. In the process of evaluation, the uncertain factors influencing the project duration plan are regarded as random variables, and probability distributions of the random variables are assumed to be normal distribution, which is determined using two parameters evaluated from the mean and standard deviation of statistical data.

However, in practical engineering, distributions of the uncertain influence factors are usually unknown. Only the statistical data of the uncertain influencing factors can be gotten, and the first four moments (mean, standard deviation, skewness, and kurtosis) of the uncertain influencing factors according to the statistical data can be easily determined. Under such circumstance, the evaluation results of the construction schedule plan will be unreasonable under the assumption that probability distributions of random variables are submitted to the normal distribution. Thus, a reasonable method to analysis the reliability of the construction schedule plan model is needed.

The objective of this paper is to propose a method by fully utilizing the statistical data to analysis the reliability of project duration plan.

4.1.2 Organization

This paper is organized as follows: First, the project duration plan based on BIM model was formulated, then the reliability model of the project duration was proposed. Next, the reliability analysis of the project duration plan model was conducted using the cubic normal transformation, in which the parameters can be determined in terms of the first four moments. Finally, the accuracy of the proposed methodology for the reliability analysis.



4.2 Formulating the reliability model of project duration under single risk factor

4.2.1 Influencing risk factors of project duration

In practical engineering, the risk factors affecting the progress of the project are complex and diverse, with different classification methods. In summary, it can be integrated into eight categories: human factors, material supply, capital supply, technical level, organization and management level, construction conditions and environment, design changes, and risk factors [7], as shown in Table 4-1.

In the above-mentioned classification table of factors affecting construction progress, some influencing factors are related. For example, human factors can also be reflected in material supply factors, capital supply factors, technical level factors, management level factors, etc. In this section, only the reliability assessment of the project duration under the influence of individual risk factor is considered. In the follow-up study, multiple risk factors and their correlation will be considered according to the situation.

Table 4-1 Classification of risk factors affecting project duration

Serial number	Classification of influencing factors	Main performance
1	the quality of people	owner, designer, contractor, supervisor, government, etc.
2	goods supply	supply timeliness, quality, quantity, specifications, performance compliance
3	capital supply	project payment
4	technical level	construction plan and construction technology, safety measures
5	organizational management level	construction organization and management level, coordination management level of all parties
6	construction condition and environment	natural environment, social environment
7	design changes	design changes
8	risk factors	political risk, economic risk, technical risk, natural disasters, etc.

4.2.2 Formulating the performance function of project duration under single risk factor

After making the construction schedule plan based on BIM model, the schedule performance function Z is [6]:

$$Z = T_M - T_Y \tag{4-1}$$

where T_M indicates the target schedule plan and T_Y indicates the analysis schedule plan.

The values of Z have three conditions:

$Z > 0$ is the reliable state of analysis schedule, which indicates actual schedule ahead of schedule;

$Z < 0$ is the failure state to analysis schedule, which indicates the actual schedule delay;

$Z = 0$ is the critical limit state for analysis schedule.

The relationship between target schedule and analysis schedule, indicating that the work is completed on time.

The degree of impact of a change in an influencing factor on schedule is called the degree of impact, which is obtained by comprehensive evaluation using symbol ε(X) to represent. The relationship between target schedule and analysis schedule can be expressed as follows:

$$T_Y(\mathbf{X}) = T_M + \varepsilon(\mathbf{X})T_M = [1+\varepsilon(\mathbf{X})]T_M \tag{4-2}$$

where **X** is the influencing factors of the construction schedule.

There are many kinds of influencing factors in a project. While the reliability analysis of single influencing factor (like the supply of concrete) is rather simpler, sometimes we must evaluate the impact degree of single influencing factor to the project, therefore the reliability analysis of single influencing factor construction schedule is the aim of this study.

According to Li [6], the impact degree of single influencing factor in construction schedule ε(x) is expressed as:

$$\varepsilon(x) = \delta - ax, x \in (0, x_0) \tag{4-3}$$

where $\delta \in (0,1)$ is the prediction accuracy and varying in engineering; a is the determined value of single factor affecting the construction schedule. x is the single influencing factor, when $x \geq x_0$, $\varepsilon(x)=0$.

So that for single influencing factor the performance function Z can be expressed as:

$$Z = T_M - T_Y = -\varepsilon(x)T_M = (ax-\delta)T_M \tag{4-4}$$

4.3 Reliability analysis of project duration

In practical engineering, simple factor such as the supply of concrete, the number of workers, weather effects, is usually assumed as normal distribution random variables. However, in most cases, the distributions of the factors are unknown, but the statistical inspection data of those factors can be collected. And the first four moments (mean, standard deviation, skewness, and kurtosis) of the random variable x can be easily obtained according to the inspection data.

The probability of failure can be expressed as the following equation according to its definition:

$$\begin{aligned} P_F &= \mathrm{Prob}(Z<0) = Prob[(ax-\delta)] < 0 = Prob(x < \delta / a) \\ &= Prob(\frac{x-\mu_x}{\sigma_x} < \frac{\delta / a - \mu_x}{\sigma_x}) = Prob(x_s < \frac{\delta / a - \mu_x}{\sigma_x}) \end{aligned} \tag{4-5}$$

where $x_s=(x-\mu_x)/\sigma_x$ denotes the standardized random variable, μ_x is the mean of the statistical data, σ_x is the standard deviation of the statistical data.

Suppose the CDF and PDF of x_s are Fx_s and fx_s, respectively, then

$$P_F = Fx_s(x_s < \frac{\delta / a - \mu_x}{\sigma_x}) = \int_{-\infty}^{\frac{\delta / a - \mu_x}{\sigma_x}} fx_s(x_s)dx_s \tag{4-6}$$

That is to say, the probability of failure is the value of CDF of x_s at$\{\delta/a - \mu_x\}/\sigma_x$, which is also expressed as the arear under the PDF curve less than$\{\delta/a - \mu_x\}/\sigma_x$.

Let

$$Fx_s(x_s) = \Phi(u) \tag{4-7}$$

According to the equation above, the relationship between the standardized variable x_s and standard normal variable u can be expressed as the following function

$$x_s = S_u(u) = a_1 + a_2 u + a_3 u^2 + a_4 u^3 \tag{4-8}$$

where $Su(u)$ is a third-order polynomial of u, Φ and ϕ are the CDF and PDF of a standard normal random variable u; a_1, a_2, a_3 and a_4 are coefficients up to the first four moments of the left side of Eq. (4-8) and it is equal to those of the right side [8].

Therefore, the inverse function of the relationship between standard normal variable u the standardized variable x_s can then be expressed as

$$u = S^{-1}(x_s) \tag{4-9}$$

where the S^{1} denotes inverse function of S, and the explicit expressions of u are summarized in Table 4-2 [9].

The parameters p, q, Δ, θ, J_1^*, J_2^* and J_0 of Table 4-2 can be calculated as the following equations:

$$p = \frac{3b - a^2}{9} \tag{4-10}$$

$$q = \frac{a^2}{27} - \frac{ab}{6} - \frac{a}{2} - \frac{x_s}{2a_4}, \Delta = \sqrt{p^3 + q^2} \tag{4-11}$$

$$\theta = \arccos\left[-q/(\sqrt{-p})^3\right] \tag{4-12}$$

$$J_1^* = \sigma_x a_4(-2|p|^{3/2} + 2q + G_s / a_4) + \mu_x \tag{4-13}$$

$$J_2^* = \sigma_x a_4(2|p|^{3/2} + 2q + x_s / a_4) + \mu_x \tag{4-13}$$

$$J_0 = -(a_2^2 / 4a_3 + a_3)\delta_x + \mu_x \tag{4-14}$$

where parameters a and b are respectively defined as $a = a_3/a_4$ and $b = a_2/a_4$. It can be seen from Table 4-2 that there are six types in the cubic normal distribution, including unbounded distributions (Types I and VI), unilaterally bounded distributions (Types II, III, and V), and a bounded distribution (Type IV).

Table 4-2 Expressions of u

Parameters			u	Range of x	Type
$p \geq 0$			$-p/\sqrt[3]{\Delta - p} + \sqrt[3]{\Delta - p} - a/3$	$(-\infty, \infty)$	I
$p < 0$	$a_4 > 0$	$\alpha_{3x} \geq 0$	$2\sqrt{-p}\cos(\theta/3) - a/3$	$J_1^* < x < J_2^*$	II
			$-p/\sqrt[3]{\Delta - p} + \sqrt[3]{\Delta - p} - a/3$	$x \geq J_2^*$	
		$\alpha_{3x} < 0$	$-p/\sqrt{\Delta - p} + \sqrt{\Delta - p} - a/3$	$x \leq J_1^*$	III
			$-p/\sqrt{-p}\cos[(\theta - \pi)/3] - a/3$	$J_1^* < x < J_2^*$	
	$a_4 > 0$		$-p/\sqrt{-p}\cos[(\theta + \pi)/3] - a/3$	$J_1^* \leq x \leq J_2^*$	IV
	$a_4 = 0$	$\alpha_{3x} > 0$	$\sqrt{1/4 + (a_3/a_2)^2 + a_3 x_s / a_2} - 1/2$	$x \geq J_0$	V
		$\alpha_{3x} < 0$	$\sqrt{1/4 + (a_3/a_2)^2 + a_3 x_s / a_2} - 1/2$	$x \leq J_0$	
		$\alpha_{3x} = 0$	x_s	$(-\infty, \infty)$	VI

According to Eq. (4-9) with an explicit expression listed in Table.4-2 into Eq. (4-7), one obtains that:

$$F_{x_s}(x_s) = \Phi(u) = \Phi[S^{-1}(x_s)] \tag{4-16}$$

Therefore, the failure probability of the project duration can be expressed as

$$P_F = F_{x_s}\left(\frac{-\mu_x}{\sigma_x}\right) = \Phi\left[S^{-1}\left(\frac{-\mu_x}{\sigma_x}\right)\right] \tag{4-17}$$

4.4 Illustrative example

4.4.1 Project background

Among the factors affecting the construction schedule, the supply of concrete, which belongs to the type of goods supply, is influenced by quality of concrete mixing plant equipment, the storage of raw materials, whether the operator is operating reasonably, the traffic and other factors, having strong randomness. Thus, in the subsequent analysis, the supply of concrete is taken as an example.

To better verify the accuracy of the improved method, the project named as Building 12 of the Garden of Tian Zhong Wei Ye [6] was investigated. In the construction process, the fourth period of waterproofing and raft foundation need to work for 10 days. The corresponding BIM model is shown in Fig. 4-1.

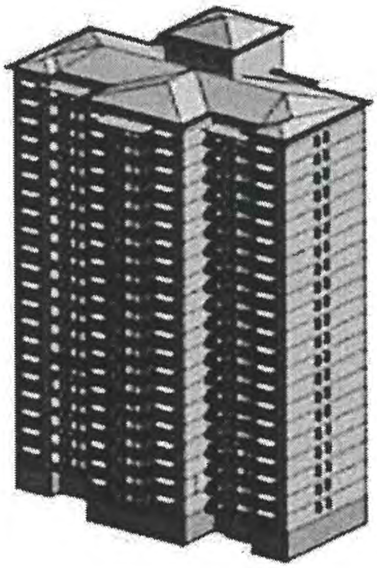


Fig. 4-1 The BIM model of Building 12 of Tian Zhong Wei Ye [6]

The target schedule of this the fourth construction stage is $T_M = 10$ days. The supply of concrete is the risk factor, which will influence the project duration. Take the supply of concrete as a random variable and the distribution of it is unknown. The past statistical data records are shown in Table 4-3 [10].

Table 4-3 The records of concrete supply

Number	Concrete supply	Number	Concrete supply
1	65.7	51	83
2	68	52	75
3	70.3	53	83
4	43	54	43
5	65	55	70

Table 4-3 Continued

6	45	56	60
7	49	57	45
8	50	58	64
9	79	59	75
10	72	60	64
11	42	61	64
12	71	62	69
13	77	63	64
14	98	64	65
15	92	65	64
16	95	66	75
17	91	67	64
18	89	68	75
19	73.5	69	75
20	64	70	63
21	74.5	71	75
22	64	72	75
23	74	73	66
24	64	74	75
25	74	75	75
26	64	76	84
27	95	77	75
28	95	78	75
29	77	79	84
30	90	80	57
31	88	81	83.5
32	88	82	84
33	92	83	87
34	86	84	83
35	90	85	84
36	81	86	83
37	92	87	68
38	104	88	83
39	103	89	74
40	75	90	73
41	85	91	84
42	84	92	81
43	73	93	72
44	73	94	72
45	73	95	63
46	83	96	48
47	84	97	99
48	73	98	81
49	84	99	81
50	83	100	81

Table 4-3 Continued

101	63	151	69
102	75	152	55
103	81	153	69
104	56	154	65
105	56	155	65
106	54	156	68
107	81	157	59
108	81	158	70
109	56	159	54
110	54	160	56
111	65	161	52
112	66	162	65
113	72	163	51
114	66	164	60
115	65	165	60
116	75	166	68
117	64	167	58
118	78	168	65
119	68	169	68
120	77	170	64
121	78	171	68
122	66	172	69
123	79	173	53
124	80	174	49
125	82	175	110
126	60	176	61
127	85	177	60
128	66	178	65
129	59	179	80
130	65	180	67
131	61	181	58
132	45	182	63
133	60	183	61
134	59	184	65
135	68	185	65
136	68	186	58
137	75	187	66
138	66	188	69
139	60	189	100
140	75	190	53
141	63	191	80
142	60	192	105
143	55	193	58
144	79	194	68
145	65	195	68
146	68	196	110
147	68	197	58
148	55	198	50
149	68	199	58
150	66	200	54

4.4.2 Reliability analysis of the fourth period

According to past statistical data records in Table 4-3 of the supply of concrete, the first four moments of the supply of commercial concrete are easy to calculate, so that according to the first four moments, we can get the coefficients from the Table A given by Zhao et al [11], as shown in Table 4-4. The detail about Table A is given in the appendix A.

Table 4-4 The first four moments of the data and coefficients of the cubic normal distribution

First four moments		Coefficients of the cubic normal distribution	
μ_x	70.7875	a_1	-0.0700
σ_x	13.1786	a_2	0.9935
α_{3x}	0.4199	a_3	0.0700
α_{4x}	3.2480	a_4	0.0005

According to Table 4-4 and Eq. (4-10), we can get the value of α_{3x}, a_4, a, b, p:

$$a = a_3/a_4 = 132.9996 \tag{4-18}$$

$$b = a_2/a_4 = 1887.7560 \tag{4-19}$$

$$\text{a3x} = 0.4199 > 0 \tag{4-20}$$

$$\text{a4} = 0.0005 > 0 \tag{4-21}$$

$$p = \frac{3b - a^2}{9} = 1336.1810 < 0 \tag{4-22}$$

So that from Table 4-2, it can be seen u belongs to Type II. With Eq.(4-11)-(4-13), the parameters q ;Δ, θ, J_1^*, J_2^* can be gotten.

$$q = \frac{a^3}{27} - \frac{ab}{6} - \frac{a}{2} - \frac{x_s}{2a_4} = 45222.3089 - \frac{(x - 70.7875)}{0.01387} \tag{4-23}$$

$$\Delta = \sqrt{p^3 + q^2} = \sqrt{(-1336.1810)^3 + q^2} \tag{4-24}$$

$$\theta = \arccos\left[-q/(\sqrt{-p})^3\right] = \arccos\left[-q/(\sqrt{1336.1810})^3\right] \tag{4-25}$$

$$J_1^* = \sigma_x a_4(-2|p|^{3/2} + 2q + x_s / a_4) + \mu_x$$
$$= 0.0069(-97685 + 2q + \frac{(x - 70.7875)}{0.0069}) + 70.7875 \tag{4-26}$$

$$J_2^* = \sigma_x a_4(2|p|^{3/2} + 2q + x_s / a_4) + \mu_x$$
$$= 0.0069(97685 + 2q + \frac{(x - 70.7875)}{0.0069}) + 70.7875 \tag{4-27}$$

The corresponding histogram is presented in Fig. 4-2, along with the fitted PDF of the normal distribution, whose mean value and standard deviation are equal to those of the statistical data, and the fitted PDF of the cubic normal distribution whose first four moments of the data and coefficients are shown in Table 4-4.

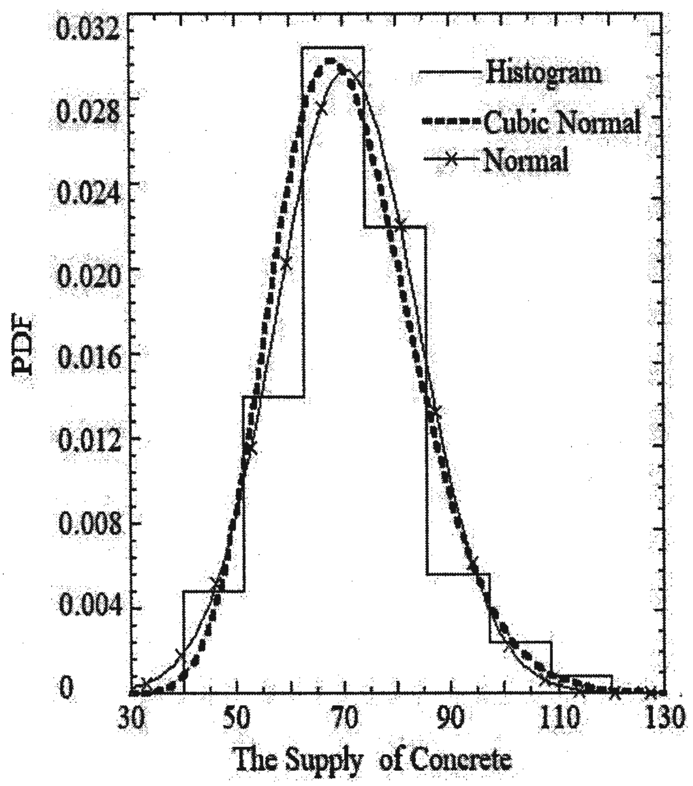


Fig. 4-2 Comparison between normal distribution and cubic normal distribution in fitting statistical data

It can be seen from Fig. 4-2 that the cubic normal distribution fits the histogram much better than the normal distribution. The influence degree $|\varepsilon(x)| < \delta$, while $\delta = 20\%$. The expression of the influence degree of commercial concrete supply is as follows:

$$\varepsilon(x) = \delta - ax = 0.2 - 0.0028x, x \in (0,71.4)$$

If the daily supply of concrete is x >71.428 m3, then $\varepsilon = 0$. The parameters $p = 3990.53$, $J_1^* =$ 3.5437×108; J_2^* = -1.3712×107, thus the type of the inverse transformation is II.

The performance function Z of construction duration is

$$Z = (ax - \delta)T_M = 0.028x - 2$$

According to Eq. (4-17), the probability of failure is

$$P_F = F_{x_s}\left(\frac{71.4286 - 70.7875}{13.1786}\right) = \Phi(-0.118) = 0.5471$$

The failure probability of construction schedule for one day delay of the fourth section of raft foundation is 0.5471.

4.5 Conclusions

In this work, the reliability analysis method of construction schedule model based on cubic normal transformation, which utilizing the first four moments of the influencing factor in practical engineering was proposed. It can provide more accurate analysis result than other existing methods to analysis the reliability of construction schedule plan. Through case analysis, the proposed analysis method using cubic normal distribution is proved to be more reliable than that using the normal distribution.

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Appendix The four parameters of a_1, a_2, a_3 and a_4 for given values of skewness α_3 and kurtosis α_4

Table A: The four parameters of a_1, a_2, a_3 and a_4 for given values of skewness α_3 and kurtosis α_4 (Cubic normal distribution)

α_3=0.00				α_3=0.05				α_3=0.10			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
2.0	0.0	1.2210	-0.0802	2.0	-0.0138	1.2224	-0.0808	2.0	-0.0280	1.2268	-0.0828
2.2	0.0	1.1478	-0.0520	2.2	-0.0116	1.1487	-0.0524	2.2	-0.0233	1.1514	-0.0535
2.4	0.0	1.0972	-0.0335	2.4	-0.0103	1.0979	-0.0338	2.4	-0.0206	1.0998	-0.0346
2.6	0.0	1.0585	-0.0199	2.6	-0.0094	1.0590	-0.0201	2.6	-0.0189	1.0605	-0.0207
2.8	0.0	1.0269	-0.0090	2.8	-0.0088	1.0273	-0.0092	2.8	-0.0177	1.0285	-0.0097
3.0	0.0	1.0000	0.0000	3.0	-0.0083	1.0004	-0.0001	3.0	-0.0167	1.0014	-0.0006
3.2	0.0	0.9765	0.0078	3.2	-0.0080	0.9768	0.0076	3.2	-0.0160	0.9777	0.0073
3.4	0.0	0.9555	0.0146	3.4	-0.0076	0.9558	0.0145	3.4	-0.0153	0.9566	0.0142
3.6	0.0	0.9365	0.0207	3.6	-0.0074	0.9368	0.0206	3.6	-0.0148	0.9375	0.0203
3.8	0.0	0.9191	0.0263	3.8	-0.0072	0.9193	0.0262	3.8	-0.0143	0.9200	0.0259
4.0	0.0	0.9030	0.0314	4.0	-0.0070	0.9032	0.0313	4.0	-0.0139	0.9038	0.0310
4.2	0.0	0.8879	0.0361	4.2	-0.0068	0.8881	0.0360	4.2	-0.0136	0.8886	0.0358
4.4	0.0	0.8738	0.0404	4.4	-0.0066	0.8739	0.0404	4.4	-0.0132	0.8744	0.0402
4.6	0.0	0.8604	0.0445	4.6	-0.0065	0.8606	0.0445	4.6	-0.0129	0.8610	0.0443
4.8	0.0	0.8477	0.0484	4.8	-0.0063	0.8479	0.0483	4.8	-0.0127	0.8483	0.0482
5.0	0.0	0.8357	0.0521	5.0	-0.0062	0.8358	0.0520	5.0	-0.0124	0.8362	0.0518
5.2	0.0	0.8241	0.0555	5.2	-0.0061	0.8243	0.0555	5.2	-0.0122	0.8247	0.0553
5.4	0.0	0.8131	0.0588	5.4	-0.0060	0.8132	0.0588	5.4	-0.0120	0.8136	0.0586
5.6	0.0	0.8025	0.0620	5.6	-0.0059	0.8026	0.0619	5.6	-0.0118	0.8029	0.0618
5.8	0.0	0.7922	0.0650	5.8	-0.0058	0.7923	0.0650	5.8	-0.0116	0.7927	0.0648
6.0	0.0	0.7824	0.0679	6.0	-0.0057	0.7825	0.0679	6.0	-0.0114	0.7828	0.0677
6.2	0.0	0.7728	0.0707	6.2	-0.0056	0.7729	0.0707	6.2	-0.0113	0.7732	0.0705
6.4	0.0	0.7636	0.0734	6.4	-0.0056	0.7637	0.0733	6.4	-0.0111	0.7639	0.0732
6.6	0.0	0.7546	0.0760	6.6	-0.0055	0.7547	0.0759	6.6	-0.0110	0.7550	0.0758
6.8	0.0	0.7459	0.0785	6.8	-0.0054	0.7460	0.0785	6.8	-0.0109	0.7462	0.0784
7.0	0.0	0.7374	0.0809	7.0	-0.0054	0.7375	0.0809	7.0	-0.0107	0.7377	0.0808
7.2	0.0	0.7291	0.0833	7.2	-0.0053	0.7292	0.0833	7.2	-0.0106	0.7295	0.0832
7.4	0.0	0.7211	0.0856	7.4	-0.0052	0.7211	0.0855	7.4	-0.0105	0.7214	0.0854
7.6	0.0	0.7132	0.0878	7.6	-0.0052	0.7133	0.0878	7.6	-0.0104	0.7135	0.0877
7.8	0.0	0.7055	0.0900	7.8	-0.0051	0.7056	0.0899	7.8	-0.0103	0.7058	0.0899
8.0	0.0	0.6980	0.0921	8.0	-0.0051	0.6981	0.0920	8.0	-0.0102	0.6983	0.0920
8.2	0.0	0.6907	0.0941	8.2	-0.0050	0.6907	0.0941	8.2	-0.0101	0.6909	0.0940
8.4	0.0	0.6835	0.0961	8.4	-0.0050	0.6835	0.0961	8.4	-0.0100	0.6837	0.0960
8.6	0.0	0.6764	0.0981	8.6	-0.0049	0.6765	0.0981	8.6	-0.0099	0.6767	0.0980
8.8	0.0	0.6695	0.1000	8.8	-0.0049	0.6695	0.1000	8.8	-0.0098	0.6697	0.0999
9.0	0.0	0.6627	0.1019	9.0	-0.0049	0.6627	0.1019	9.0	-0.0097	0.6629	0.1018

Table A: *Continued*

	α_3=0.15				α_3=0.20				α_3=0.25		
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
2.0	-0.0429	1.2343	-0.0863	2.0	-0.0590	1.2459	-0.0917	2.0	-0.0775	1.2634	-0.1000
2.2	-0.0354	1.1560	-0.0555	2.2	-0.0481	1.1627	-0.0585	2.2	-0.0617	1.1718	-0.0625
2.4	-0.0313	1.1032	-0.0360	2.4	-0.0422	1.1079	-0.0380	2.4	-0.0537	1.1143	-0.0407
2.6	-0.0286	1.0631	-0.0218	2.6	-0.0385	1.0668	-0.0233	2.6	-0.0487	1.0716	-0.0253
2.8	-0.0267	1.0306	-0.0106	2.8	-0.0358	1.0336	-0.0118	2.8	-0.0453	1.0375	-0.0134
3.0	-0.0252	1.0032	-0.0013	3.0	-0.0338	1.0057	-0.0023	3.0	-0.0426	1.0090	-0.0036
3.2	-0.0240	0.9792	0.0067	3.2	-0.0322	0.9814	0.0058	3.2	-0.0406	0.9842	0.0047
3.4	-0.0231	0.9580	0.0137	3.4	-0.0309	0.9599	0.0129	3.4	-0.0389	0.9623	0.0119
3.6	-0.0223	0.9387	0.0199	3.6	-0.0298	0.9404	0.0192	3.6	-0.0374	0.9426	0.0183
3.8	-0.0215	0.9211	0.0255	3.8	-0.0288	0.9226	0.0249	3.8	-0.0362	0.9246	0.0241
4.0	-0.0209	0.9048	0.0307	4.0	-0.0280	0.9062	0.0301	4.0	-0.0352	0.9080	0.0294
4.2	-0.0204	0.8896	0.0354	4.2	-0.0273	0.8908	0.0349	4.2	-0.0342	0.8925	0.0343
4.4	-0.0199	0.8753	0.0398	4.4	-0.0266	0.8765	0.0394	4.4	-0.0334	0.8780	0.0388
4.6	-0.0195	0.8618	0.0440	4.6	-0.0260	0.8629	0.0436	4.6	-0.0326	0.8644	0.0430
4.8	-0.0190	0.8491	0.0479	4.8	-0.0255	0.8501	0.0475	4.8	-0.0319	0.8514	0.0470
5.0	-0.0187	0.8369	0.0516	5.0	-0.0250	0.8379	0.0512	5.0	-0.0313	0.8391	0.0507
5.2	-0.0183	0.8253	0.0551	5.2	-0.0245	0.8262	0.0547	5.2	-0.0307	0.8274	0.0540
5.4	-0.0180	0.8142	0.0584	5.4	-0.0241	0.8151	0.0581	5.4	-0.0302	0.8162	0.0576
5.6	-0.0177	0.8035	0.0616	5.6	-0.0237	0.8043	0.0613	5.6	-0.0297	0.8054	0.0608
5.8	-0.0175	0.7932	0.0646	5.8	-0.0233	0.7940	0.0643	5.8	-0.0292	0.7950	0.0639
6.0	-0.0172	0.7833	0.0675	6.0	-0.0230	0.7841	0.0672	6.0	-0.0288	0.7850	0.0669
6.2	-0.0170	0.7737	0.0703	6.2	-0.0226	0.7744	0.0701	6.2	-0.0284	0.7754	0.0697
6.4	-0.0167	0.7644	0.0730	6.4	-0.0223	0.7651	0.0728	6.4	-0.0280	0.7660	0.0724
6.6	-0.0165	0.7554	0.0757	6.6	-0.0220	0.7561	0.0754	6.6	-0.0276	0.7569	0.0751
6.8	-0.0163	0.7467	0.0782	6.8	-0.0218	0.7473	0.0779	6.8	-0.0273	0.7481	0.0776
7.0	-0.0161	0.7382	0.0806	7.0	-0.0215	0.7388	0.0804	7.0	-0.0269	0.7396	0.0801
7.2	-0.0159	0.7299	0.0830	7.2	-0.0213	0.7305	0.0828	7.2	-0.0266	0.7312	0.0825
7.4	-0.0157	0.7218	0.0853	7.4	-0.0210	0.7224	0.0851	7.4	-0.0263	0.7231	0.0848
7.6	-0.0156	0.7139	0.0875	7.6	-0.0208	0.7145	0.0873	7.6	-0.0260	0.7152	0.0870
7.8	-0.0154	0.7062	0.0897	7.8	-0.0206	0.7067	0.0895	7.8	-0.0258	0.7074	0.0892
8.0	-0.0153	0.6987	0.0918	8.0	-0.0204	0.6992	0.0916	8.0	-0.0255	0.6999	0.0914
8.2	-0.0151	0.6913	0.0939	8.2	-0.0202	0.6918	0.0937	8.2	-0.0253	0.6924	0.0934
8.4	-0.0150	0.6841	0.0959	8.4	-0.0200	0.6846	0.0957	8.4	-0.0250	0.6852	0.0955
8.6	-0.0148	0.6770	0.0979	8.6	-0.0198	0.6775	0.0977	8.6	-0.0248	0.6781	0.0975
8.8	-0.0147	0.6701	0.0998	8.8	-0.0196	0.6705	0.0996	8.8	-0.0246	0.6711	0.0994
9.0	-0.0146	0.6633	0.1017	9.0	-0.0195	0.6637	0.1015	9.0	-0.0244	0.6643	0.1013

Table A: *Continued*

α_3=0.30				α_3=0.35				α_3=0.40			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
2.2	-0.0766	1.1836	-0.0678	2.2	-0.0936	1.1989	-0.0749	2.2	-0.1136	1.2192	-0.0848
2.4	-0.0660	1.1225	-0.0443	2.4	-0.0793	1.1328	-0.0488	2.4	-0.0941	1.1455	-0.0545
2.6	-0.0595	1.0778	-0.0279	2.6	-0.0709	1.0855	-0.0312	2.6	-0.0832	1.0949	-0.0352
2.8	-0.0550	1.0425	-0.0154	2.8	-0.0653	1.0486	-0.0179	2.8	-0.0761	1.0560	-0.0210
3.0	-0.0517	1.0131	-0.0053	3.0	-0.0611	1.0182	-0.0074	3.0	-0.0710	1.0242	-0.0099
3.2	-0.0491	0.9878	0.0033	3.2	-0.0579	0.9921	0.0015	3.2	-0.0671	0.9972	-0.0006
3.4	-0.0470	0.9654	0.0107	3.4	-0.0554	0.9692	0.0092	3.4	-0.0640	0.9736	0.0074
3.6	-0.0452	0.9454	0.0172	3.6	-0.0532	0.9487	0.0159	3.6	-0.0614	0.9526	0.0143
3.8	-0.0437	0.9271	0.0231	3.8	-0.0514	0.9301	0.0219	3.8	-0.0592	0.9336	0.0205
4.0	-0.0424	0.9102	0.0285	4.0	-0.0498	0.9129	0.0274	4.0	-0.0574	0.9161	0.0262
4.2	-0.0413	0.8946	0.0335	4.2	-0.0484	0.8971	0.0325	4.2	-0.0557	0.9000	0.0313
4.4	-0.0402	0.8799	0.0380	4.4	-0.0472	0.8822	0.0371	4.4	-0.0543	0.8849	0.0361
4.6	-0.0393	0.8661	0.0423	4.6	-0.0461	0.8683	0.0415	4.6	-0.0530	0.8708	0.0405
4.8	-0.0384	0.8531	0.0463	4.8	-0.0451	0.8551	0.0455	4.8	-0.0518	0.8574	0.0446
5.0	-0.0377	0.8407	0.0501	5.0	-0.0441	0.8426	0.0494	5.0	-0.0507	0.8447	0.0485
5.2	-0.0370	0.8289	0.0537	5.2	-0.0433	0.8306	0.0530	5.2	-0.0497	0.8327	0.0522
5.4	-0.0363	0.8176	0.0571	5.4	-0.0425	0.8192	0.0564	5.4	-0.0488	0.8212	0.0557
5.6	-0.0357	0.8067	0.0603	5.6	-0.0418	0.8083	0.0597	5.6	-0.0480	0.8101	0.0590
5.8	-0.0351	0.7963	0.0634	5.8	-0.0411	0.7978	0.0629	5.8	-0.0472	0.7995	0.0622
6.0	-0.0346	0.7862	0.0664	6.0	-0.0405	0.7877	0.0659	6.0	-0.0465	0.7893	0.0652
6.2	-0.0341	0.7765	0.0693	6.2	-0.0399	0.7779	0.0687	6.2	-0.0458	0.7795	0.0681
6.4	-0.0336	0.7671	0.0720	6.4	-0.0394	0.7684	0.0715	6.4	-0.0451	0.7699	0.0709
6.6	-0.0332	0.7580	0.0747	6.6	-0.0388	0.7592	0.0742	6.6	-0.0445	0.7607	0.0736
6.8	-0.0328	0.7491	0.0772	6.8	-0.0383	0.7503	0.0768	6.8	-0.0439	0.7517	0.0762
7.0	-0.0324	0.7405	0.0797	7.0	-0.0379	0.7417	0.0793	7.0	-0.0434	0.7430	0.0788
7.2	-0.0320	0.7322	0.0821	7.2	-0.0374	0.7333	0.0817	7.2	-0.0429	0.7346	0.0812
7.4	-0.0316	0.7240	0.0844	7.4	-0.0370	0.7251	0.0840	7.4	-0.0424	0.7263	0.0836
7.6	-0.0313	0.7160	0.0867	7.6	-0.0366	0.7171	0.0863	7.6	-0.0419	0.7183	0.0858
7.8	-0.0310	0.7083	0.0889	7.8	-0.0362	0.7093	0.0885	7.8	-0.0415	0.7104	0.0881
8.0	-0.0307	0.7007	0.0911	8.0	-0.0358	0.7017	0.0907	8.0	-0.0411	0.7028	0.0902
8.2	-0.0304	0.6932	0.0931	8.2	-0.0355	0.6942	0.0928	8.2	-0.0406	0.6953	0.0924
8.4	-0.0301	0.6860	0.0952	8.4	-0.0351	0.6869	0.0948	8.4	-0.0403	0.6880	0.0944
8.6	-0.0298	0.6788	0.0972	8.6	-0.0348	0.6797	0.0968	8.6	-0.0399	0.6808	0.0964
8.8	-0.0295	0.6719	0.0991	8.8	-0.0345	0.6727	0.0988	8.8	-0.0395	0.6737	0.0984
9.0	-0.0293	0.6650	0.1010	9.0	-0.0342	0.6658	0.1007	9.0	-0.0392	0.6668	0.1003
9.2	-0.0290	0.6583	0.1029	9.2	-0.0339	0.6591	0.1026	9.2	-0.0388	0.6600	0.1022

Table A: *Continued*

α_3=0.45				α_3=0.50				α_3=0.55			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
2.2	-0.1406	1.2523	-0.1014	2.4	-0.1311	1.1811	-0.0714	2.6	-0.1297	1.1370	-0.0543
2.4	-0.1110	1.1613	-0.0618	2.6	-0.1120	1.1202	-0.0465	2.8	-0.1147	1.0881	-0.0350
2.6	-0.0967	1.1064	-0.0402	2.8	-0.1005	1.0755	-0.0294	3.0	-0.1048	1.0500	-0.0208
2.8	-0.0877	1.0649	-0.0248	3.0	-0.0926	1.0400	-0.0165	3.2	-0.0977	1.0187	-0.0095
3.0	-0.0814	1.0314	-0.0129	3.2	-0.0868	1.0104	-0.0060	3.4	-0.0923	0.9920	-0.0002
3.2	-0.0767	1.0033	-0.0031	3.4	-0.0824	0.9850	0.0027	3.6	-0.0880	0.9687	0.0078
3.4	-0.0730	0.9789	0.0052	3.6	-0.0787	0.9625	0.0103	3.8	-0.0844	0.9478	0.0148
3.6	-0.0699	0.9572	0.0125	3.8	-0.0757	0.9424	0.0170	4.0	-0.0814	0.9290	0.0210
3.8	-0.0673	0.9377	0.0189	4.0	-0.0731	0.9241	0.0230	4.2	-0.0788	0.9116	0.0267
4.0	-0.0651	0.9198	0.0247	4.2	-0.0709	0.9072	0.0284	4.4	-0.0766	0.8956	0.0318
4.2	-0.0632	0.9034	0.0300	4.4	-0.0689	0.8916	0.0334	4.6	-0.0745	0.8806	0.0366
4.4	-0.0615	0.8880	0.0348	4.6	-0.0672	0.8769	0.0381	4.8	-0.0727	0.8666	0.0410
4.6	-0.0600	0.8736	0.0394	4.8	-0.0656	0.8631	0.0424	5.0	-0.0711	0.8533	0.0451
4.8	-0.0586	0.8601	0.0436	5.0	-0.0642	0.8501	0.0464	5.2	-0.0696	0.8407	0.0490
5.0	-0.0574	0.8473	0.0475	5.2	-0.0629	0.8377	0.0502	5.4	-0.0683	0.8288	0.0527
5.2	-0.0562	0.8351	0.0513	5.4	-0.0617	0.8259	0.0538	5.6	-0.0670	0.8173	0.0562
5.4	-0.0552	0.8234	0.0548	5.6	-0.0606	0.8146	0.0573	5.8	-0.0659	0.8063	0.0595
5.6	-0.0542	0.8122	0.0582	5.8	-0.0595	0.8038	0.0605	6.0	-0.0648	0.7958	0.0627
5.8	-0.0533	0.8015	0.0614	6.0	-0.0586	0.7934	0.0636	6.2	-0.0638	0.7856	0.0657
6.0	-0.0525	0.7912	0.0645	6.2	-0.0577	0.7833	0.0666	6.4	-0.0629	0.7758	0.0686
6.2	-0.0517	0.7813	0.0674	6.4	-0.0569	0.7736	0.0695	6.6	-0.0620	0.7664	0.0714
6.4	-0.0510	0.7717	0.0703	6.6	-0.0561	0.7643	0.0723	6.8	-0.0611	0.7572	0.0741
6.6	-0.0503	0.7624	0.0730	6.8	-0.0553	0.7552	0.0749	7.0	-0.0604	0.7483	0.0767
6.8	-0.0496	0.7533	0.0756	7.0	-0.0546	0.7463	0.0775	7.2	-0.0596	0.7396	0.0792
7.0	-0.0490	0.7446	0.0782	7.2	-0.0540	0.7377	0.0800	7.4	-0.0589	0.7312	0.0817
7.2	-0.0484	0.7361	0.0806	7.4	-0.0533	0.7294	0.0824	7.6	-0.0582	0.7230	0.0840
7.4	-0.0478	0.7278	0.0830	7.6	-0.0527	0.7212	0.0847	7.8	-0.0576	0.7150	0.0863
7.6	-0.0473	0.7197	0.0853	7.8	-0.0522	0.7133	0.0870	8.0	-0.0570	0.7072	0.0886
7.8	-0.0468	0.7118	0.0876	8.0	-0.0516	0.7055	0.0892	8.2	-0.0564	0.6995	0.0907
8.0	-0.0463	0.7041	0.0898	8.2	-0.0511	0.6980	0.0913	8.4	-0.0558	0.6921	0.0928
8.2	-0.0458	0.6965	0.0919	8.4	-0.0506	0.6905	0.0934	8.6	-0.0553	0.6848	0.0949
8.4	-0.0454	0.6892	0.0940	8.6	-0.0501	0.6833	0.0955	8.8	-0.0548	0.6776	0.0969
8.6	-0.0450	0.6819	0.0960	8.8	-0.0496	0.6762	0.0975	9.0	-0.0543	0.6706	0.0989
8.8	-0.0446	0.6749	0.0980	9.0	-0.0492	0.6692	0.0994	9.2	-0.0538	0.6637	0.1008
9.0	-0.0442	0.6679	0.0999	9.2	-0.0488	0.6624	0.1013	9.4	-0.0533	0.6570	0.1027
9.2	-0.0438	0.6611	0.1018	9.4	-0.0483	0.6556	0.1032	9.6	-0.0529	0.6503	0.1045

Table A: *Continued*

α_3=0.60				α_3=0.65				α_3=0.70			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
2.6	-0.1512	1.1576	-0.0645	2.6	-0.1793	1.1848	-0.0790	2.6	-0.2462	1.2756	-0.1300
2.8	-0.1310	1.1032	-0.0420	2.8	-0.1503	1.1214	-0.0507	2.8	-0.1744	1.1436	-0.0622
3.0	-0.1183	1.0619	-0.0260	3.0	-0.1337	1.0759	-0.0324	3.0	-0.1517	1.0926	-0.0403
3.2	-0.1095	1.0284	-0.0137	3.2	-0.1225	1.0397	-0.0186	3.2	-0.1372	1.0530	-0.0246
3.4	-0.1029	1.0002	-0.0036	3.4	-0.1144	1.0096	-0.0077	3.4	-0.1271	1.0206	-0.0125
3.6	-0.0978	0.9757	0.0049	3.6	-0.1082	0.9838	0.0015	3.6	-0.1196	0.9931	-0.0025
3.8	-0.0936	0.9540	0.0122	3.8	-0.1033	0.9611	0.0093	3.8	-0.1136	0.9692	0.0059
4.0	-0.0901	0.9345	0.0188	4.0	-0.0991	0.9408	0.0162	4.0	-0.1088	0.9479	0.0132
4.2	-0.0871	0.9166	0.0246	4.2	-0.0957	0.9223	0.0223	4.2	-0.1047	0.9286	0.0197
4.4	-0.0844	0.9002	0.0300	4.4	-0.0927	0.9053	0.0279	4.4	-0.1012	0.9110	0.0256
4.6	-0.0821	0.8848	0.0349	4.6	-0.0900	0.8895	0.0330	4.6	-0.0982	0.8948	0.0309
4.8	-0.0801	0.8705	0.0395	4.8	-0.0877	0.8748	0.0377	4.8	-0.0955	0.8796	0.0358
5.0	-0.0782	0.8569	0.0437	5.0	-0.0856	0.8610	0.0421	5.0	-0.0932	0.8654	0.0403
5.2	-0.0766	0.8441	0.0477	5.2	-0.0837	0.8479	0.0462	5.2	-0.0910	0.8521	0.0445
5.4	-0.0750	0.8319	0.0515	5.4	-0.0820	0.8355	0.0501	5.4	-0.0891	0.8394	0.0485
5.6	-0.0736	0.8203	0.0550	5.6	-0.0804	0.8236	0.0537	5.6	-0.0873	0.8273	0.0523
5.8	-0.0723	0.8092	0.0584	5.8	-0.0789	0.8123	0.0572	5.8	-0.0857	0.8158	0.0558
6.0	-0.0711	0.7985	0.0616	6.0	-0.0776	0.8015	0.0605	6.0	-0.0842	0.8048	0.0592
6.2	-0.0700	0.7882	0.0647	6.2	-0.0763	0.7910	0.0636	6.2	-0.0828	0.7942	0.0624
6.4	-0.0690	0.7783	0.0677	6.4	-0.0751	0.7810	0.0666	6.4	-0.0815	0.7840	0.0655
6.6	-0.0680	0.7687	0.0705	6.6	-0.0740	0.7713	0.0695	6.6	-0.0803	0.7741	0.0684
6.8	-0.0670	0.7594	0.0733	6.8	-0.0730	0.7619	0.0723	6.8	-0.0791	0.7646	0.0712
7.0	-0.0662	0.7504	0.0759	7.0	-0.0720	0.7528	0.0750	7.0	-0.0780	0.7554	0.0740
7.2	-0.0653	0.7417	0.0784	7.2	-0.0711	0.7440	0.0776	7.2	-0.0770	0.7465	0.0766
7.4	-0.0645	0.7332	0.0809	7.4	-0.0702	0.7354	0.0801	7.4	-0.0760	0.7378	0.0791
7.6	-0.0638	0.7249	0.0833	7.6	-0.0694	0.7270	0.0825	7.6	-0.0751	0.7293	0.0816
7.8	-0.0631	0.7168	0.0856	7.8	-0.0686	0.7189	0.0848	7.8	-0.0743	0.7211	0.0840
8.0	-0.0624	0.7090	0.0879	8.0	-0.0679	0.7109	0.0871	8.0	-0.0734	0.7131	0.0863
8.2	-0.0617	0.7013	0.0901	8.2	-0.0672	0.7032	0.0893	8.2	-0.0726	0.7053	0.0885
8.4	-0.0611	0.6937	0.0922	8.4	-0.0665	0.6956	0.0915	8.4	-0.0719	0.6976	0.0907
8.6	-0.0605	0.6864	0.0943	8.6	-0.0658	0.6882	0.0936	8.6	-0.0712	0.6902	0.0928
8.8	-0.0599	0.6792	0.0963	8.8	-0.0652	0.6809	0.0956	8.8	-0.0705	0.6828	0.0949
9.0	-0.0594	0.6721	0.0983	9.0	-0.0646	0.6738	0.0976	9.0	-0.0698	0.6757	0.0969
9.2	-0.0589	0.6652	0.1002	9.2	-0.0640	0.6669	0.0996	9.2	-0.0692	0.6687	0.0989
9.4	-0.0584	0.6584	0.1021	9.4	-0.0634	0.6600	0.1015	9.4	-0.0685	0.6618	0.1008
9.6	-0.0579	0.6518	0.1039	9.6	-0.0629	0.6533	0.1033	9.6	-0.0680	0.6550	0.1027

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□

Table A: *Continued*

α_3=0.75				α_3=0.80				α_3=0.85			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
2.8	-0.2076	1.1733	-0.0791	3.0	-0.2021	1.1369	-0.0638	3.0	-0.2505	1.1780	-0.0897
3.0	-0.1736	1.1125	-0.0503	3.2	-0.1750	1.0872	-0.0412	3.2	-0.2013	1.1093	-0.0532
3.	-0.1544	1.0687	-0.0320	3.4	-0.1580	1.0483	-0.0252	3.4	-0.1778	1.0659	-0.0339
3.4	-0.1414	1.0334	-0.0182	3.6	-0.1461	1.0162	-0.0128	3.6	-0.1622	1.0307	-0.0195
3.6	-0.1320	1.0038	-0.0072	3.8	-0.1372	0.9889	-0.0026	3.8	-0.1510	1.0011	-0.0081
3.8	-0.1248	0.9784	0.0020	4.0	-0.1302	0.9651	0.0059	4.0	-0.1425	0.9755	0.0013
4.0	-0.1191	0.9559	0.0098	4.2	-0.1246	0.9439	0.0133	4.2	-0.1357	0.9530	0.0094
4.2	-0.1143	0.9358	0.0167	4.4	-0.1198	0.9247	0.0199	4.4	-0.1301	0.9329	0.0164
4.4	-0.1103	0.9175	0.0229	4.6	-0.1158	0.9072	0.0258	4.6	-0.1254	0.9145	0.0227
4.6	-0.1068	0.9006	0.0285	4.8	-0.1123	0.8910	0.0311	4.8	-0.1214	0.8977	0.0283
4.8	-0.1037	0.8850	0.0336	5.0	-0.1093	0.8759	0.0360	5.0	-0.1179	0.8820	0.0335
5.0	-0.1010	0.8704	0.0383	5.2	-0.1065	0.8618	0.0406	5.2	-0.1148	0.8674	0.0383
5.2	-0.0986	0.8567	0.0427	5.4	-0.1040	0.8484	0.0448	5.4	-0.1120	0.8537	0.0427
5.4	-0.0964	0.8437	0.0468	5.6	-0.1018	0.8358	0.0488	5.6	-0.1095	0.8407	0.0468
5.6	-0.0944	0.8314	0.0506	5.8	-0.0998	0.8238	0.0526	5.8	-0.1072	0.8284	0.0507
5.8	-0.0926	0.8196	0.0543	6.0	-0.0979	0.8123	0.0562	6.0	-0.1051	0.8166	0.0544
6.0	-0.0909	0.8084	0.0577	6.2	-0.0962	0.8013	0.0595	6.2	-0.1032	0.8054	0.0579
6.2	-0.0894	0.7976	0.0610	6.4	-0.0946	0.7908	0.0628	6.4	-0.1014	0.7947	0.0612
6.4	-0.0879	0.7872	0.0642	6.6	-0.0931	0.7806	0.0658	6.6	-0.0997	0.7843	0.0644
6.6	-0.0866	0.7772	0.0672	6.8	-0.0917	0.7708	0.0688	6.8	-0.0982	0.7743	0.0674
6.8	-0.0853	0.7676	0.0701	7.0	-0.090	0.7613	0.0716	7.0	-0.0967	0.7647	0.0703
7.0	-0.0841	0.7582	0.0728	7.2	-0.0891	0.7521	0.0743	7.2	-0.0954	0.7554	0.0731
7.2	-0.0830	0.7492	0.0755	7.4	-0.0880	0.7432	0.0770	7.4	-0.0941	0.7463	0.0757
7.4	-0.0819	0.7404	0.0781	7.6	-0.0869	0.7346	0.0795	7.6	-0.0929	0.7376	0.0783
7.6	-0.0809	0.7319	0.0806	7.8	-0.0858	0.7262	0.0820	7.8	-0.0917	0.7290	0.0808
7.8	-0.0800	0.7235	0.0830	8.0	-0.0848	0.7180	0.0844	8.0	-0.0907	0.7208	0.0833
8.0	-0.0791	0.7154	0.0853	8.2	-0.0839	0.7100	0.0867	8.2	-0.0896	0.7127	0.0856
8.2	-0.0782	0.7075	0.0876	8.4	-0.0830	0.7022	0.0889	8.4	-0.0886	0.7048	0.0879
8.4	-0.0774	0.6998	0.0898	8.6	-0.0821	0.6946	0.0911	8.6	-0.0877	0.6971	0.0901
8.6	-0.0766	0.6923	0.0920	8.8	-0.0813	0.6871	0.0932	8.8	-0.0868	0.6896	0.0923
8.8	-0.0758	0.6849	0.0941	9.0	-0.0805	0.6799	0.0953	9.0	-0.0859	0.6822	0.0944
9.0	-0.0751	0.6777	0.0961	9.2	-0.0797	0.6727	0.0973	9.2	-0.0851	0.6750	0.0964
9.2	-0.0744	0.6706	0.0981	9.4	-0.0790	0.6657	0.0993	9.4	-0.0843	0.6679	0.0984
9.4	-0.0737	0.6637	0.1001	9.6	-0.0783	0.6589	0.1012	9.6	-0.0836	0.6610	0.1004
9.6	-0.0731	0.6569	0.1020	9.8	-0.0776	0.6521	0.1031	9.8	-0.0828	0.6542	0.1023
9.8	-0.0725	0.6502	0.1038	10.0	-0.0770	0.6455	0.1049	10.0	-0.0821	0.6476	0.1041

Table A: *Continued*

α_3=0.90				α_3=0.95				α_3=1.00			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
3.2	-0.2380	1.1372	-0.0705	3.4	-0.2362	1.1113	-0.0601	3.6	-0.2375	1.0904	-0.0524
3.4	-0.2027	1.0866	-0.0450	3.6	-0.2056	1.0674	-0.0384	3.8	-0.2096	1.0507	-0.0330
3.6	-0.1815	1.0476	-0.0278	3.8	-0.1860	1.0316	-0.0228	4.0	-0.1910	1.0175	-0.0186
3.8	-0.1670	1.0152	-0.0147	4.0	-0.1721	1.0014	-0.0106	4.2	-0.1776	0.9891	-0.0071
4.0	-0.1563	0.9875	-0.0041	4.2	-0.1617	0.9754	-0.0006	4.4	-0.1674	0.9643	0.0024
4.2	-0.1480	0.9634	0.0048	4.4	-0.1536	0.9525	0.0078	4.6	-0.1594	0.9424	0.0105
4.4	-0.1413	0.9420	0.0124	4.6	-0.1470	0.9320	0.0152	4.8	-0.1528	0.9227	0.0176
4.6	-0.1358	0.9227	0.0192	4.8	-0.1414	0.9134	0.0217	5.0	-0.1472	0.9048	0.0239
4.8	-0.1311	0.9051	0.0252	5.0	-0.1367	0.8963	0.0275	5.2	-0.1424	0.8882	0.0295
5.0	-0.1270	0.8888	0.0307	5.2	-0.1326	0.8806	0.0328	5.4	-0.1383	0.8729	0.0347
5.2	-0.1234	0.8737	0.0357	5.4	-0.1290	0.8658	0.0377	5.6	-0.1346	0.8585	0.0395
5.4	-0.1203	0.8595	0.0403	5.6	-0.1258	0.8520	0.0422	5.8	-0.1313	0.8450	0.0439
5.6	-0.1174	0.8461	0.0446	5.8	-0.1229	0.8389	0.0464	6.0	-0.1284	0.8322	0.0480
5.8	-0.1149	0.8334	0.0487	6.0	-0.1203	0.8266	0.0504	6.2	-0.1257	0.8201	0.0519
6.0	-0.1125	0.8214	0.0525	6.2	-0.1179	0.8148	0.0541	6.4	-0.1233	0.8085	0.0556
6.2	-0.1104	0.8099	0.0561	6.4	-0.1157	0.8035	0.0576	6.6	-0.1210	0.7974	0.0590
6.4	-0.1084	0.7989	0.0595	6.6	-0.1137	0.7927	0.0610	6.8	-0.1189	0.7868	0.0624
6.6	-0.1066	0.7883	0.0627	6.8	-0.1118	0.7823	0.0642	7.0	-0.1170	0.7766	0.0655
6.8	-0.1049	0.7782	0.0658	7.0	-0.1100	0.7723	0.0672	7.2	-0.1152	0.7667	0.0685
7.0	-0.1033	0.7683	0.0688	7.2	-0.1084	0.7626	0.0702	7.4	-0.1135	0.7572	0.0714
7.2	-0.1018	0.7589	0.0717	7.4	-0.1068	0.7533	0.0730	7.6	-0.1119	0.7480	0.0742
7.4	-0.1004	0.7497	0.0744	7.6	-0.1054	0.7443	0.0757	7.8	-0.1104	0.7391	0.0769
7.6	-0.0991	0.7408	0.0771	7.8	-0.1040	0.7355	0.0783	8.0	-0.1090	0.7304	0.0794
7.8	-0.0978	0.7321	0.0796	8.0	-0.1027	0.7269	0.0808	8.2	-0.1076	0.7220	0.0819
8.0	-0.0966	0.7237	0.0821	8.2	-0.1015	0.7186	0.0833	8.4	-0.1064	0.7138	0.0843
8.2	-0.0955	0.7155	0.0845	8.4	-0.1003	0.7106	0.0856	8.6	-0.1051	0.7058	0.0867
8.4	-0.0944	0.7076	0.0868	8.6	-0.0992	0.7027	0.0879	8.8	-0.1040	0.6980	0.0890
8.6	-0.0934	0.6998	0.0891	8.8	-0.0981	0.6950	0.0901	9.0	-0.1029	0.6903	0.0912
8.8	-0.0924	0.6922	0.0912	9.0	-0.0971	0.6874	0.0923	9.2	-0.1018	0.6829	0.0933
9.0	-0.0915	0.6847	0.0934	9.2	-0.0962	0.6801	0.0944	9.4	-0.1008	0.6756	0.0954
9.2	-0.0906	0.6774	0.0955	9.4	-0.0952	0.6729	0.0965	9.6	-0.0999	0.6685	0.0974
9.4	-0.0897	0.6703	0.0975	9.6	-0.0943	0.6658	0.0985	9.8	-0.0989	0.6615	0.0994
9.6	-0.0889	0.6633	0.0995	9.8	-0.0935	0.6589	0.1004	10.0	-0.0980	0.6546	0.1014
9.8	-0.0881	0.6565	0.1014	10.0	-0.0926	0.6521	0.1024	10.2	-0.0972	0.6479	0.1033
10.0	-0.0873	0.6497	0.1033	10.2	-0.0918	0.6454	0.1042	10.4	-0.0964	0.6413	0.1051
10.2	-0.0866	0.6431	0.1051	10.4	-0.0911	0.6389	0.1061	10.6	-0.0956	0.6348	0.1069

Table A: *Continued*

α_3=1.05			
α_4	a_1=-a_3	a_2	a_4
3.6	-0.2895	1.1224	-0.0765
3.8	-0.2407	1.0727	-0.0465
4.0	-0.2144	1.0361	-0.0286
4.2	-0.1965	1.0049	-0.0150
4.4	-0.1834	0.9780	-0.0041
4.6	-0.1734	0.9543	0.0050
4.8	-0.1653	0.9332	0.0128
5.0	-0.1587	0.9142	0.0197
5.2	-0.1531	0.8968	0.0258
5.4	-0.1482	0.8807	0.0314
5.6	-0.1440	0.8657	0.0364
5.8	-0.1403	0.8516	0.0411
6.0	-0.1369	0.8384	0.0454
6.2	-0.1339	0.8259	0.0495
6.4	-0.1312	0.8140	0.0533
6.6	-0.1286	0.8026	0.0569
6.8	-0.1263	0.7917	0.0604
7.0	-0.1242	0.7812	0.0636
7.2	-0.1222	0.7711	0.0667
7.4	-0.1203	0.7614	0.0697
7.6	-0.1186	0.7520	0.0726
7.8	-0.1169	0.7429	0.0753
8.0	-0.1154	0.7341	0.0779
8.2	-0.1139	0.7255	0.0805
8.4	-0.1125	0.7172	0.0830
8.6	-0.1112	0.7091	0.0854
8.8	-0.1100	0.7012	0.0877
9.0	-0.1088	0.6934	0.0899
9.2	-0.1076	0.6859	0.0921
9.4	-0.1065	0.6785	0.0942
9.6	-0.1055	0.6713	0.0963
9.8	-0.1045	0.6642	0.0983
10.0	-0.1035	0.6573	0.1003
10.2	-0.1026	0.6505	0.1022
10.4	-0.1017	0.6438	0.1041
10.6	-0.1009	0.6372	0.1060

α_3=1.10			
α_4	a_1=-a_3	a_2	a_4
3.8	-0.2874	1.0995	-0.0669
4.0	-0.2453	1.0574	-0.0417
4.2	-0.2201	1.0233	-0.0250
4.4	-0.2025	0.9937	-0.0120
4.6	-0.1895	0.9680	-0.0015
4.8	-0.1795	0.9452	0.0073
5.0	-0.1714	0.9248	0.0149
5.2	-0.1647	0.9063	0.0216
5.4	-0.1590	0.8893	0.0276
5.6	-0.1541	0.8736	0.0330
5.8	-0.1498	0.8590	0.0380
6.0	-0.1460	0.8452	0.0426
6.2	-0.1426	0.8322	0.0468
6.4	-0.1395	0.8199	0.0508
6.6	-0.1366	0.8082	0.0546
6.8	-0.1341	0.7970	0.0582
7.0	-0.1317	0.7862	0.0615
7.2	-0.1295	0.7759	0.0648
7.4	-0.1274	0.7660	0.0678
7.6	-0.1255	0.7564	0.0708
7.8	-0.1237	0.7471	0.0736
8.0	-0.1220	0.7381	0.0763
8.2	-0.1204	0.7294	0.0790
8.4	-0.1189	0.7209	0.0815
8.6	-0.1174	0.7127	0.0839
8.8	-0.1161	0.7046	0.0863
9.0	-0.1148	0.6968	0.0886
9.2	-0.1136	0.6891	0.0908
9.4	-0.1124	0.6816	0.0930
9.6	-0.1112	0.6743	0.0951
9.8	-0.1102	0.6671	0.0972
10.0	-0.1091	0.6601	0.0992
10.2	-0.1081	0.6532	0.1011
10.4	-0.1072	0.6465	0.1031
10.6	-0.1062	0.6399	0.1049
10.8	-0.1054	0.6334	0.1068

α_3=1.20			
α_4	a_1=-a_3	a_2	a_4
4.2	-0.2960	1.0670	-0.0565
4.4	-0.2578	1.0322	-0.0350
4.6	-0.2333	1.0016	-0.0196
4.8	-0.2158	0.9746	-0.0075
5.0	-0.2027	0.9506	0.0025
5.2	-0.1924	0.9293	0.0109
5.4	-0.1841	0.9100	0.0182
5.6	-0.1771	0.8923	0.0246
5.8	-0.1712	0.8761	0.0304
6.0	-0.1661	0.8610	0.0357
6.2	-0.1616	0.8469	0.0405
6.4	-0.1576	0.8335	0.0450
6.6	-0.1540	0.8210	0.0492
6.8	-0.1507	0.8090	0.0531
7.0	-0.1477	0.7976	0.0568
7.2	-0.1450	0.7867	0.0603
7.4	-0.1424	0.7762	0.0636
7.6	-0.1401	0.7661	0.0668
7.8	-0.1379	0.7564	0.0698
8.0	-0.1359	0.7470	0.0727
8.2	-0.1340	0.7379	0.0755
8.4	-0.1322	0.7291	0.0781
8.6	-0.1304	0.7206	0.0807
8.8	-0.1288	0.7122	0.0832
9.0	-0.1273	0.7041	0.0856
9.2	-0.1259	0.6962	0.0880
9.4	-0.1245	0.6885	0.0902
9.6	-0.1231	0.6810	0.0925
9.8	-0.1219	0.6736	0.0946
10.0	-0.1207	0.6664	0.0967
10.2	-0.1195	0.6593	0.0987
10.4	-0.1184	0.6524	0.1007
10.6	-0.1173	0.6456	0.1026
10.8	-0.1163	0.6390	0.1045
11.0	-0.1153	0.6324	0.1064
11.2	-0.1144	0.6260	0.1082

Table A: *Continued*

α_3=1.30				α_3=1.40				α_3=1.50			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=- a_3	a_2	a_4
4.6	-0.3132	1.0424	-0.0516	5.0	-0.3386	1.0218	-0.0509	5.4	-0.3772	1.0023	-0.0559
4.8	-0.2746	1.0122	-0.0313	5.2	-0.2959	0.9958	-0.0302	5.6	-0.3233	0.9814	-0.0318
5.0	-0.2491	0.9843	-0.0164	5.4	-0.2681	0.9703	-0.0152	5.8	-0.2911	0.9587	-0.0161
5.2	-0.2309	0.9590	-0.0046	5.6	-0.2482	0.9464	-0.0034	6.0	-0.2683	0.9362	-0.0037
5.4	-0.2171	0.9364	0.0051	5.8	-0.2332	0.9247	0.0064	6.2	-0.2514	0.9152	0.0064
5.6	-0.2063	0.9160	0.0133	6.0	-0.2215	0.9049	0.0147	6.4	-0.2383	0.8959	0.0150
5.8	-0.1975	0.8974	0.0205	6.2	-0.2120	0.8869	0.0219	6.6	-0.2277	0.8782	0.0223
6.0	-0.1902	0.8804	0.0268	6.4	-0.2041	0.8703	0.0282	6.8	-0.2190	0.8619	0.0288
6.2	-0.1839	0.8647	0.0325	6.6	-0.1973	0.8550	0.0339	7.0	-0.2116	0.8468	0.0346
6.4	-0.1785	0.8500	0.0377	6.8	-0.1915	0.8406	0.0391	7.2	-0.2052	0.8326	0.0398
6.6	-0.1737	0.8363	0.0425	7.0	-0.1864	0.8271	0.0438	7.4	-0.1996	0.8193	0.0446
6.8	-0.1695	0.8233	0.0469	7.2	-0.1818	0.8144	0.0482	7.6	-0.1947	0.8067	0.0490
7.0	-0.1656	0.8110	0.0510	7.4	-0.1777	0.8023	0.0524	7.8	-0.1902	0.7948	0.0532
7.2	-0.1622	0.7993	0.0549	7.6	-0.1740	0.7908	0.0562	8.0	-0.1862	0.7834	0.0570
7.4	-0.1590	0.7882	0.0585	7.8	-0.1706	0.7798	0.0598	8.2	-0.1825	0.7725	0.0607
7.6	-0.1561	0.7775	0.0620	8.0	-0.1675	0.7693	0.0633	8.4	-0.1792	0.7621	0.0641
7.8	-0.1534	0.7672	0.0653	8.2	-0.1646	0.7592	0.0665	8.6	-0.1761	0.7521	0.0674
8.0	-0.1509	0.7573	0.0684	8.4	-0.1619	0.7495	0.0696	8.8	-0.1732	0.7425	0.0705
8.2	-0.1486	0.7478	0.0714	8.6	-0.1594	0.7400	0.0726	9.0	-0.1705	0.7331	0.0735
8.4	-0.1464	0.7386	0.0742	8.8	-0.1571	0.7309	0.0755	9.2	-0.1680	0.7241	0.0764
8.6	-0.1443	0.7296	0.0770	9.0	-0.1549	0.7221	0.0782	9.4	-0.1657	0.7154	0.0791
8.8	-0.1424	0.7209	0.0796	9.2	-0.1528	0.7135	0.0808	9.6	-0.1635	0.7069	0.0817
9.0	-0.1406	0.7125	0.0822	9.4	-0.1509	0.7052	0.0834	9.8	-0.1614	0.6986	0.0843
9.2	-0.1389	0.7043	0.0847	9.6	-0.1491	0.6971	0.0858	10.0	-0.1594	0.6906	0.0867
9.4	-0.1373	0.6963	0.0871	9.8	-0.1473	0.6892	0.0882	10.2	-0.1576	0.6828	0.0891
9.6	-0.1357	0.6885	0.0894	10.0	-0.1457	0.6815	0.0905	10.4	-0.1558	0.6751	0.0914
9.8	-0.1342	0.6809	0.0916	10.2	-0.1441	0.6740	0.0928	10.6	-0.1541	0.6677	0.0937
10.0	-0.1328	0.6735	0.0938	10.4	-0.1426	0.6666	0.0949	10.8	-0.1525	0.6604	0.0958
10.2	-0.1315	0.6662	0.0959	10.6	-0.1411	0.6594	0.0971	11.0	-0.1509	0.6532	0.0979
10.4	-0.1302	0.6591	0.0980	10.8	-0.1397	0.6524	0.0991	11.2	-0.1495	0.6463	0.1000
10.6	-0.1289	0.6521	0.1000	11.0	-0.1384	0.6455	0.1011	11.4	-0.1480	0.6394	0.1020
10.8	-0.1277	0.6453	0.1020	11.2	-0.1372	0.6387	0.1031	11.6	-0.1467	0.6327	0.1039
11.0	-0.1266	0.6386	0.1039	11.4	-0.1359	0.6321	0.1050	11.8	-0.1454	0.6261	0.1059
11.2	-0.1255	0.6320	0.1058	11.6	-0.1348	0.6256	0.1069	12.0	-0.1441	0.6196	0.1077
11.4	-0.1244	0.6255	0.1076	11.8	-0.1336	0.6192	0.1087	12.2	-0.1429	0.6133	0.1095
11.6	-0.1234	0.6192	0.1094	12.0	-0.1325	0.6129	0.1105	12.4	-0.1417	0.6070	0.1113

Table A: *Continued*

α_3=1.60			
α_4	a_1=-a_3	a_2	a_4
6.0	-0.3603	0.9665	-0.0370
6.2	-0.3202	0.9484	-0.0194
6.4	-0.2926	0.9279	-0.0059
6.6	-0.2726	0.9077	0.0050
6.8	-0.2573	0.8888	0.0140
7.0	-0.2452	0.8713	0.0217
7.2	-0.2353	0.8551	0.0284
7.4	-0.2270	0.8400	0.0344
7.6	-0.2199	0.8259	0.0398
7.8	-0.2137	0.8127	0.0447
8.0	-0.2083	0.8002	0.0493
8.2	-0.2034	0.7883	0.0535
8.4	-0.1990	0.7770	0.0574
8.6	-0.1950	0.7662	0.0611
8.8	-0.1913	0.7559	0.0646
9.0	-0.1880	0.7459	0.0679
9.2	-0.1849	0.7363	0.0710
9.4	-0.1820	0.7270	0.0740
9.6	-0.1793	0.7181	0.0769
9.8	-0.1768	0.7094	0.0797
10.0	-0.1744	0.7009	0.0823
10.2	-0.1721	0.6927	0.0849
10.4	-0.1700	0.6847	0.0873
10.6	-0.1680	0.6770	0.0897
10.8	-0.1661	0.6694	0.0920
11.0	-0.1643	0.6619	0.0943
11.2	-0.1626	0.6547	0.0965
11.4	-0.1609	0.6476	0.0986
11.6	-0.1593	0.6407	0.1006
11.8	-0.1578	0.6338	0.1027
12.0	-0.1563	0.6272	0.1046
12.2	-0.1549	0.6206	0.1065
12.4	-0.1536	0.6142	0.1084
12.6	-0.1523	0.6078	0.1102
12.8	-0.1511	0.6016	0.1120
13.0	-0.1499	0.5955	0.1137

α_3=1.70			
α_4	a_1=-a_3	a_2	a_4
6.4	-0.4200	0.9446	-0.0498
6.6	-0.3594	0.9366	-0.0259
6.8	-0.3236	0.9202	-0.0105
7.0	-0.2983	0.9016	0.0017
7.2	-0.2796	0.8833	0.0116
7.4	-0.2651	0.8659	0.0199
7.6	-0.2535	0.8498	0.0271
7.8	-0.2440	0.8347	0.0334
8.0	-0.2359	0.8206	0.0391
8.2	-0.2289	0.8073	0.0442
8.4	-0.2228	0.7948	0.0489
8.6	-0.2174	0.7829	0.0532
8.8	-0.2125	0.7716	0.0573
9.0	-0.2081	0.7608	0.0610
9.2	-0.2041	0.7505	0.0646
9.4	-0.2004	0.7405	0.0680
9.6	-0.1970	0.7310	0.0712
9.8	-0.1939	0.7217	0.0742
10.0	-0.1909	0.7128	0.0771
10.2	-0.1882	0.7041	0.0799
10.4	-0.1856	0.6957	0.0826
10.6	-0.1832	0.6875	0.0852
10.8	-0.1809	0.6795	0.0877
11.0	-0.1787	0.6718	0.0901
11.2	-0.1767	0.6642	0.0924
11.4	-0.1747	0.6568	0.0947
11.6	-0.1729	0.6496	0.0969
11.8	-0.1711	0.6425	0.0990
12.0	-0.1694	0.6356	0.1011
12.2	-0.1678	0.6288	0.1031
12.4	-0.1662	0.6221	0.1051
12.6	-0.1647	0.6156	0.1070
12.8	-0.1633	0.6092	0.1088
13.0	-0.1619	0.6029	0.1107
13.2	-0.1605	0.5967	0.1125
13.4	-0.1592	0.5906	0.1142

α_3=1.80			
α_4	a_1=-a_3	a_2	a_4
7.0	-0.4192	0.9155	-0.0386
7.2	-0.3662	0.9102	-0.0186
7.4	-0.3317	0.8958	-0.0042
7.6	-0.3071	0.8789	0.0072
7.8	-0.2888	0.8620	0.0166
8.0	-0.2746	0.8459	0.0246
8.2	-0.2632	0.8307	0.0314
8.4	-0.2536	0.8165	0.0375
8.6	-0.2456	0.8032	0.0429
8.8	-0.2386	0.7906	0.0478
9.0	-0.2324	0.7786	0.0524
9.2	-0.2269	0.7673	0.0565
9.4	-0.2220	0.7564	0.0605
9.6	-0.2176	0.7460	0.0641
9.8	-0.2135	0.7360	0.0676
10.0	-0.2097	0.7264	0.0709
10.2	-0.2063	0.7172	0.0740
10.4	-0.2031	0.7082	0.0770
10.6	-0.2001	0.6995	0.0798
10.8	-0.1973	0.6911	0.0826
11.0	-0.1946	0.6829	0.0852
11.2	-0.1921	0.6750	0.0877
11.4	-0.1898	0.6672	0.0902
11.6	-0.1876	0.6596	0.0925
11.8	-0.1855	0.6522	0.0948
12.0	-0.1834	0.6450	0.0970
12.2	-0.1815	0.6380	0.0992
12.4	-0.1797	0.6310	0.1013
12.6	-0.1779	0.6243	0.1033
12.8	-0.1763	0.6176	0.1053
13.0	-0.1747	0.6111	0.1072
13.2	-0.1731	0.6047	0.1091
13.4	-0.1716	0.5984	0.1110
13.6	-0.1702	0.5922	0.1128
13.8	-0.1688	0.5861	0.1145
14.0	-0.1675	0.5801	0.1163

Table A: *Continued*

α_3=0.90				α_3=0.95				α_3=1.00			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
3.2	-0.2380	1.1372	-0.0705	3.4	-0.2362	1.1113	-0.0601	3.6	-0.2375	1.0904	-0.0524
3.4	-0.2027	1.0866	-0.0450	3.6	-0.2056	1.0674	-0.0384	3.8	-0.2096	1.0507	-0.0330
3.6	-0.1815	1.0476	-0.0278	3.8	-0.1860	1.0316	-0.0228	4.0	-0.1910	1.0175	-0.0186
3.8	-0.1670	1.0152	-0.0147	4.0	-0.1721	1.0014	-0.0106	4.2	-0.1776	0.9891	-0.0071
4.0	-0.1563	0.9875	-0.0041	4.2	-0.1617	0.9754	-0.0006	4.4	-0.1674	0.9643	0.0024
4.2	-0.1480	0.9634	0.0048	4.4	-0.1536	0.9525	0.0078	4.6	-0.1594	0.9424	0.0105
4.4	-0.1413	0.9420	0.0124	4.6	-0.1470	0.9320	0.0152	4.8	-0.1528	0.9227	0.0176
4.6	-0.1358	0.9227	0.0192	4.8	-0.1414	0.9134	0.0217	5.0	-0.1472	0.9048	0.0239
4.8	-0.1311	0.9051	0.0252	5.0	-0.1367	0.8963	0.0275	5.2	-0.1424	0.8882	0.0295
5.0	-0.1270	0.8888	0.0307	5.2	-0.1326	0.8806	0.0328	5.4	-0.1383	0.8729	0.0347
5.2	-0.1234	0.8737	0.0357	5.4	-0.1290	0.8658	0.0377	5.6	-0.1346	0.8585	0.0395
5.4	-0.1203	0.8595	0.0403	5.6	-0.1258	0.8520	0.0422	5.8	-0.1313	0.8450	0.0439
5.6	-0.1174	0.8461	0.0446	5.8	-0.1229	0.8389	0.0464	6.0	-0.1284	0.8322	0.0480
5.8	-0.1149	0.8334	0.0487	6.0	-0.1203	0.8266	0.0504	6.2	-0.1257	0.8201	0.0519
6.0	-0.1125	0.8214	0.0525	6.2	-0.1179	0.8148	0.0541	6.4	-0.1233	0.8085	0.0556
6.2	-0.1104	0.8099	0.0561	6.4	-0.1157	0.8035	0.0576	6.6	-0.1210	0.7974	0.0590
6.4	-0.1084	0.7989	0.0595	6.6	-0.1137	0.7927	0.0610	6.8	-0.1189	0.7868	0.0624
6.6	-0.1066	0.7883	0.0627	6.8	-0.1118	0.7823	0.0642	7.0	-0.1170	0.7766	0.0655
6.8	-0.1049	0.7782	0.0658	7.0	-0.1100	0.7723	0.0672	7.2	-0.1152	0.7667	0.0685
7.0	-0.1033	0.7683	0.0688	7.2	-0.1084	0.7626	0.0702	7.4	-0.1135	0.7572	0.0714
7.2	-0.1018	0.7589	0.0717	7.4	-0.1068	0.7533	0.0730	7.6	-0.1119	0.7480	0.0742
7.4	-0.1004	0.7497	0.0744	7.6	-0.1054	0.7443	0.0757	7.8	-0.1104	0.7391	0.0769
7.6	-0.0991	0.7408	0.0771	7.8	-0.1040	0.7355	0.0783	8.0	-0.1090	0.7304	0.0794
7.8	-0.0978	0.7321	0.0796	8.0	-0.1027	0.7269	0.0808	8.2	-0.1076	0.7220	0.0819
8.0	-0.0966	0.7237	0.0821	8.2	-0.1015	0.7186	0.0833	8.4	-0.1064	0.7138	0.0843
8.2	-0.0955	0.7155	0.0845	8.4	-0.1003	0.7106	0.0856	8.6	-0.1051	0.7058	0.0867
8.4	-0.0944	0.7076	0.0868	8.6	-0.0992	0.7027	0.0879	8.8	-0.1040	0.6980	0.0890
8.6	-0.0934	0.6998	0.0891	8.8	-0.0981	0.6950	0.0901	9.0	-0.1029	0.6903	0.0912
8.8	-0.0924	0.6922	0.0912	9.0	-0.0971	0.6874	0.0923	9.2	-0.1018	0.6829	0.0933
9.0	-0.0915	0.6847	0.0934	9.2	-0.0962	0.6801	0.0944	9.4	-0.1008	0.6756	0.0954
9.2	-0.0906	0.6774	0.0955	9.4	-0.0952	0.6729	0.0965	9.6	-0.0999	0.6685	0.0974
9.4	-0.0897	0.6703	0.0975	9.6	-0.0943	0.6658	0.0985	9.8	-0.0989	0.6615	0.0994
9.6	-0.0889	0.6633	0.0995	9.8	-0.0935	0.6589	0.1004	10.0	-0.0980	0.6546	0.1014
9.8	-0.0881	0.6565	0.1014	10.0	-0.0926	0.6521	0.1024	10.2	-0.0972	0.6479	0.1033
10.0	-0.0873	0.6497	0.1033	10.2	-0.0918	0.6454	0.1042	10.4	-0.0964	0.6413	0.1051
10.2	-0.0866	0.6431	0.1051	10.4	-0.0911	0.6389	0.1061	10.6	-0.0956	0.6348	0.1069

Table A: *Continued*

α_3=1.05				α_3=1.10				α_3=1.20			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
3.6	-0.2895	1.1224	-0.0765	3.8	-0.2874	1.0995	-0.0669	4.2	-0.2960	1.0670	-0.0565
3.8	-0.2407	1.0727	-0.0465	4.0	-0.2453	1.0574	-0.0417	4.4	-0.2578	1.0322	-0.0350
4.0	-0.2144	1.0361	-0.0286	4.2	-0.2201	1.0233	-0.0250	4.6	-0.2333	1.0016	-0.0196
4.2	-0.1965	1.0049	-0.0150	4.4	-0.2025	0.9937	-0.0120	4.8	-0.2158	0.9746	-0.0075
4.4	-0.1834	0.9780	-0.0041	4.6	-0.1895	0.9680	-0.0015	5.0	-0.2027	0.9506	0.0025
4.6	-0.1734	0.9543	0.0050	4.8	-0.1795	0.9452	0.0073	5.2	-0.1924	0.9293	0.0109
4.8	-0.1653	0.9332	0.0128	5.0	-0.1714	0.9248	0.0149	5.4	-0.1841	0.9100	0.0182
5.0	-0.1587	0.9142	0.0197	5.2	-0.1647	0.9063	0.0216	5.6	-0.1771	0.8923	0.0246
5.2	-0.1531	0.8968	0.0258	5.4	-0.1590	0.8893	0.0276	5.8	-0.1712	0.8761	0.0304
5.4	-0.1482	0.8807	0.0314	5.6	-0.1541	0.8736	0.0330	6.0	-0.1661	0.8610	0.0357
5.6	-0.1440	0.8657	0.0364	5.8	-0.1498	0.8590	0.0380	6.2	-0.1616	0.8469	0.0405
5.8	-0.1403	0.8516	0.0411	6.0	-0.1460	0.8452	0.0426	6.4	-0.1576	0.8335	0.0450
6.0	-0.1369	0.8384	0.0454	6.2	-0.1426	0.8322	0.0468	6.6	-0.1540	0.8210	0.0492
6.2	-0.1339	0.8259	0.0495	6.4	-0.1395	0.8199	0.0508	6.8	-0.1507	0.8090	0.0531
6.4	-0.1312	0.8140	0.0533	6.6	-0.1366	0.8082	0.0546	7.0	-0.1477	0.7976	0.0568
6.6	-0.1286	0.8026	0.0569	6.8	-0.1341	0.7970	0.0582	7.2	-0.1450	0.7867	0.0603
6.8	-0.1263	0.7917	0.0604	7.0	-0.1317	0.7862	0.0615	7.4	-0.1424	0.7762	0.0636
7.0	-0.1242	0.7812	0.0636	7.2	-0.1295	0.7759	0.0648	7.6	-0.1401	0.7661	0.0668
7.2	-0.1222	0.7711	0.0667	7.4	-0.1274	0.7660	0.0678	7.8	-0.1379	0.7564	0.0698
7.4	-0.1203	0.7614	0.0697	7.6	-0.1255	0.7564	0.0708	8.0	-0.1359	0.7470	0.0727
7.6	-0.1186	0.7520	0.0726	7.8	-0.1237	0.7471	0.0736	8.2	-0.1340	0.7379	0.0755
7.8	-0.1169	0.7429	0.0753	8.0	-0.1220	0.7381	0.0763	8.4	-0.1322	0.7291	0.0781
8.0	-0.1154	0.7341	0.0779	8.2	-0.1204	0.7294	0.0790	8.6	-0.1304	0.7206	0.0807
8.2	-0.1139	0.7255	0.0805	8.4	-0.1189	0.7209	0.0815	8.8	-0.1288	0.7122	0.0832
8.4	-0.1125	0.7172	0.0830	8.6	-0.1174	0.7127	0.0839	9.0	-0.1273	0.7041	0.0856
8.6	-0.1112	0.7091	0.0854	8.8	-0.1161	0.7046	0.0863	9.2	-0.1259	0.6962	0.0880
8.8	-0.1100	0.7012	0.0877	9.0	-0.1148	0.6968	0.0886	9.4	-0.1245	0.6885	0.0902
9.0	-0.1088	0.6934	0.0899	9.2	-0.1136	0.6891	0.0908	9.6	-0.1231	0.6810	0.0925
9.2	-0.1076	0.6859	0.0921	9.4	-0.1124	0.6816	0.0930	9.8	-0.1219	0.6736	0.0946
9.4	-0.1065	0.6785	0.0942	9.6	-0.1112	0.6743	0.0951	10.0	-0.1207	0.6664	0.0967
9.6	-0.1055	0.6713	0.0963	9.8	-0.1102	0.6671	0.0972	10.2	-0.1195	0.6593	0.0987
9.8	-0.1045	0.6642	0.0983	10.0	-0.1091	0.6601	0.0992	10.4	-0.1184	0.6524	0.1007
10.0	-0.1035	0.6573	0.1003	10.2	-0.1081	0.6532	0.1011	10.6	-0.1173	0.6456	0.1026
10.2	-0.1026	0.6505	0.1022	10.4	-0.1072	0.6465	0.1031	10.8	-0.1163	0.6390	0.1045
10.4	-0.1017	0.6438	0.1041	10.6	-0.1062	0.6399	0.1049	11.0	-0.1153	0.6324	0.1064
10.6	-0.1009	0.6372	0.1060	10.8	-0.1054	0.6334	0.1068	11.2	-0.1144	0.6260	0.1082

Table A: *Continued*

α_3=1.30				α_3=1.40				α_3=1.50			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4	α_4	a_1=- a_3	a_2	a_4
4.6	-0.3132	1.0424	-0.0516	5.0	-0.3386	1.0218	-0.0509	5.4	-0.3772	1.0023	-0.0559
4.8	-0.2746	1.0122	-0.0313	5.2	-0.2959	0.9958	-0.0302	5.6	-0.3233	0.9814	-0.0318
5.0	-0.2491	0.9843	-0.0164	5.4	-0.2681	0.9703	-0.0152	5.8	-0.2911	0.9587	-0.0161
5.2	-0.2309	0.9590	-0.0046	5.6	-0.2482	0.9464	-0.0034	6.0	-0.2683	0.9362	-0.0037
5.4	-0.2171	0.9364	0.0051	5.8	-0.2332	0.9247	0.0064	6.2	-0.2514	0.9152	0.0064
5.6	-0.2063	0.9160	0.0133	6.0	-0.2215	0.9049	0.0147	6.4	-0.2383	0.8959	0.0150
5.8	-0.1975	0.8974	0.0205	6.2	-0.2120	0.8869	0.0219	6.6	-0.2277	0.8782	0.0223
6.0	-0.1902	0.8804	0.0268	6.4	-0.2041	0.8703	0.0282	6.8	-0.2190	0.8619	0.0288
6.2	-0.1839	0.8647	0.0325	6.6	-0.1973	0.8550	0.0339	7.0	-0.2116	0.8468	0.0346
6.4	-0.1785	0.8500	0.0377	6.8	-0.1915	0.8406	0.0391	7.2	-0.2052	0.8326	0.0398
6.6	-0.1737	0.8363	0.0425	7.0	-0.1864	0.8271	0.0438	7.4	-0.1996	0.8193	0.0446
6.8	-0.1695	0.8233	0.0469	7.2	-0.1818	0.8144	0.0482	7.6	-0.1947	0.8067	0.0490
7.0	-0.1656	0.8110	0.0510	7.4	-0.1777	0.8023	0.0524	7.8	-0.1902	0.7948	0.0532
7.2	-0.1622	0.7993	0.0549	7.6	-0.1740	0.7908	0.0562	8.0	-0.1862	0.7834	0.0570
7.4	-0.1590	0.7882	0.0585	7.8	-0.1706	0.7798	0.0598	8.2	-0.1825	0.7725	0.0607
7.6	-0.1561	0.7775	0.0620	8.0	-0.1675	0.7693	0.0633	8.4	-0.1792	0.7621	0.0641
7.8	-0.1534	0.7672	0.0653	8.2	-0.1646	0.7592	0.0665	8.6	-0.1761	0.7521	0.0674
8.0	-0.1509	0.7573	0.0684	8.4	-0.1619	0.7495	0.0696	8.8	-0.1732	0.7425	0.0705
8.2	-0.1486	0.7478	0.0714	8.6	-0.1594	0.7400	0.0726	9.0	-0.1705	0.7331	0.0735
8.4	-0.1464	0.7386	0.0742	8.8	-0.1571	0.7309	0.0755	9.2	-0.1680	0.7241	0.0764
8.6	-0.1443	0.7296	0.0770	9.0	-0.1549	0.7221	0.0782	9.4	-0.1657	0.7154	0.0791
8.8	-0.1424	0.7209	0.0796	9.2	-0.1528	0.7135	0.0808	9.6	-0.1635	0.7069	0.0817
9.0	-0.1406	0.7125	0.0822	9.4	-0.1509	0.7052	0.0834	9.8	-0.1614	0.6986	0.0843
9.2	-0.1389	0.7043	0.0847	9.6	-0.1491	0.6971	0.0858	10.0	-0.1594	0.6906	0.0867
9.4	-0.1373	0.6963	0.0871	9.8	-0.1473	0.6892	0.0882	10.2	-0.1576	0.6828	0.0891
9.6	-0.1357	0.6885	0.0894	10.0	-0.1457	0.6815	0.0905	10.4	-0.1558	0.6751	0.0914
9.8	-0.1342	0.6809	0.0916	10.2	-0.1441	0.6740	0.0928	10.6	-0.1541	0.6677	0.0937
10.0	-0.1328	0.6735	0.0938	10.4	-0.1426	0.6666	0.0949	10.8	-0.1525	0.6604	0.0958
10.2	-0.1315	0.6662	0.0959	10.6	-0.1411	0.6594	0.0971	11.0	-0.1509	0.6532	0.0979
10.4	-0.1302	0.6591	0.0980	10.8	-0.1397	0.6524	0.0991	11.2	-0.1495	0.6463	0.1000
10.6	-0.1289	0.6521	0.1000	11.0	-0.1384	0.6455	0.1011	11.4	-0.1480	0.6394	0.1020
10.8	-0.1277	0.6453	0.1020	11.2	-0.1372	0.6387	0.1031	11.6	-0.1467	0.6327	0.1039
11.0	-0.1266	0.6386	0.1039	11.4	-0.1359	0.6321	0.1050	11.8	-0.1454	0.6261	0.1059
11.2	-0.1255	0.6320	0.1058	11.6	-0.1348	0.6256	0.1069	12.0	-0.1441	0.6196	0.1077
11.4	-0.1244	0.6255	0.1076	11.8	-0.1336	0.6192	0.1087	12.2	-0.1429	0.6133	0.1095
11.6	-0.1234	0.6192	0.1094	12.0	-0.1325	0.6129	0.1105	12.4	-0.1417	0.6070	0.1113

Table A: *Continued*

α_3=1.60			
α_4	a_1=-a_3	a_2	a_4
6.0	-0.3603	0.9665	-0.0370
6.2	-0.3202	0.9484	-0.0194
6.4	-0.2926	0.9279	-0.0059
6.6	-0.2726	0.9077	0.0050
6.8	-0.2573	0.8888	0.0140
7.0	-0.2452	0.8713	0.0217
7.2	-0.2353	0.8551	0.0284
7.4	-0.2270	0.8400	0.0344
7.6	-0.2199	0.8259	0.0398
7.8	-0.2137	0.8127	0.0447
8.0	-0.2083	0.8002	0.0493
8.2	-0.2034	0.7883	0.0535
8.4	-0.1990	0.7770	0.0574
8.6	-0.1950	0.7662	0.0611
8.8	-0.1913	0.7559	0.0646
9.0	-0.1880	0.7459	0.0679
9.2	-0.1849	0.7363	0.0710
9.4	-0.1820	0.7270	0.0740
9.6	-0.1793	0.7181	0.0769
9.8	-0.1768	0.7094	0.0797
10.0	-0.1744	0.7009	0.0823
10.2	-0.1721	0.6927	0.0849
10.4	-0.1700	0.6847	0.0873
10.6	-0.1680	0.6770	0.0897
10.8	-0.1661	0.6694	0.0920
11.0	-0.1643	0.6619	0.0943
11.2	-0.1626	0.6547	0.0965
11.4	-0.1609	0.6476	0.0986
11.6	-0.1593	0.6407	0.1006
11.8	-0.1578	0.6338	0.1027
12.0	-0.1563	0.6272	0.1046
12.2	-0.1549	0.6206	0.1065
12.4	-0.1536	0.6142	0.1084
12.6	-0.1523	0.6078	0.1102
12.8	-0.1511	0.6016	0.1120
13.0	-0.1499	0.5955	0.1137

α_3=1.70			
α_4	a_1=-a_3	a_2	a_4
6.4	-0.4200	0.9446	-0.0498
6.6	-0.3594	0.9366	-0.0259
6.8	-0.3236	0.9202	-0.0105
7.0	-0.2983	0.9016	0.0017
7.2	-0.2796	0.8833	0.0116
7.4	-0.2651	0.8659	0.0199
7.6	-0.2535	0.8498	0.0271
7.8	-0.2440	0.8347	0.0334
8.0	-0.2359	0.8206	0.0391
8.2	-0.2289	0.8073	0.0442
8.4	-0.2228	0.7948	0.0489
8.6	-0.2174	0.7829	0.0532
8.8	-0.2125	0.7716	0.0573
9.0	-0.2081	0.7608	0.0610
9.2	-0.2041	0.7505	0.0646
9.4	-0.2004	0.7405	0.0680
9.6	-0.1970	0.7310	0.0712
9.8	-0.1939	0.7217	0.0742
10.0	-0.1909	0.7128	0.0771
10.2	-0.1882	0.7041	0.0799
10.4	-0.1856	0.6957	0.0826
10.6	-0.1832	0.6875	0.0852
10.8	-0.1809	0.6795	0.0877
11.0	-0.1787	0.6718	0.0901
11.2	-0.1767	0.6642	0.0924
11.4	-0.1747	0.6568	0.0947
11.6	-0.1729	0.6496	0.0969
11.8	-0.1711	0.6425	0.0990
12.0	-0.1694	0.6356	0.1011
12.2	-0.1678	0.6288	0.1031
12.4	-0.1662	0.6221	0.1051
12.6	-0.1647	0.6156	0.1070
12.8	-0.1633	0.6092	0.1088
13.0	-0.1619	0.6029	0.1107
13.2	-0.1605	0.5967	0.1125
13.4	-0.1592	0.5906	0.1142

α_3=1.80			
α_4	a_1=-a_3	a_2	a_4
7.0	-0.4192	0.9155	-0.0386
7.2	-0.3662	0.9102	-0.0186
7.4	-0.3317	0.8958	-0.0042
7.6	-0.3071	0.8789	0.0072
7.8	-0.2888	0.8620	0.0166
8.0	-0.2746	0.8459	0.0246
8.2	-0.2632	0.8307	0.0314
8.4	-0.2536	0.8165	0.0375
8.6	-0.2456	0.8032	0.0429
8.8	-0.2386	0.7906	0.0478
9.0	-0.2324	0.7786	0.0524
9.2	-0.2269	0.7673	0.0565
9.4	-0.2220	0.7564	0.0605
9.6	-0.2176	0.7460	0.0641
9.8	-0.2135	0.7360	0.0676
10.0	-0.2097	0.7264	0.0709
10.2	-0.2063	0.7172	0.0740
10.4	-0.2031	0.7082	0.0770
10.6	-0.2001	0.6995	0.0798
10.8	-0.1973	0.6911	0.0826
11.0	-0.1946	0.6829	0.0852
11.2	-0.1921	0.6750	0.0877
11.4	-0.1898	0.6672	0.0902
11.6	-0.1876	0.6596	0.0925
11.8	-0.1855	0.6522	0.0948
12.0	-0.1834	0.6450	0.0970
12.2	-0.1815	0.6380	0.0992
12.4	-0.1797	0.6310	0.1013
12.6	-0.1779	0.6243	0.1033
12.8	-0.1763	0.6176	0.1053
13.0	-0.1747	0.6111	0.1072
13.2	-0.1731	0.6047	0.1091
13.4	-0.1716	0.5984	0.1110
13.6	-0.1702	0.5922	0.1128
13.8	-0.1688	0.5861	0.1145
14.0	-0.1675	0.5801	0.1163

Table A: *Continued*

α_3=1.90				α_3=2.00			
α_4	a_1=-a_3	a_2	a_4	α_4	a_1=-a_3	a_2	a_4
7.6	-0.4336	0.8861	-0.0335	8.4	-0.4001	0.8616	-0.0126
7.8	-0.3795	0.8858	-0.0142	8.6	-0.3609	0.8539	0.0020
8.0	-0.3440	0.8740	-0.0001	8.8	-0.3337	0.8407	0.0134
8.2	-0.3188	0.8588	0.0111	9.0	-0.3137	0.8263	0.0227
8.4	-0.3002	0.8432	0.0203	9.2	-0.2985	0.8121	0.0305
8.6	-0.2857	0.8281	0.0280	9.4	-0.2862	0.7985	0.0371
8.8	-0.2740	0.8138	0.0347	9.6	-0.2761	0.7855	0.0430
9.0	-0.2643	0.8002	0.0406	9.8	-0.2675	0.7733	0.0483
9.2	-0.2561	0.7875	0.0459	10.0	-0.2600	0.7616	0.0531
9.4	-0.2489	0.7754	0.0508	10.2	-0.2535	0.7505	0.0575
9.6	-0.2426	0.7639	0.0552	10.4	-0.2477	0.7398	0.0615
9.8	-0.2370	0.7530	0.0593	10.6	-0.2424	0.7296	0.0653
10.0	-0.2320	0.7425	0.0631	10.8	-0.2377	0.7199	0.0689
10.2	-0.2274	0.7324	0.0667	11.0	-0.2334	0.7104	0.0723
10.4	-0.2232	0.7227	0.0701	11.2	-0.2294	0.7013	0.0755
10.6	-0.2194	0.7134	0.0734	11.4	-0.2257	0.6925	0.0785
10.8	-0.2158	0.7044	0.0764	11.6	-0.2223	0.6840	0.0814
11.0	-0.2125	0.6957	0.0794	11.8	-0.2191	0.6757	0.0841
11.2	-0.2094	0.6872	0.0822	12.0	-0.2161	0.6677	0.0868
11.4	-0.2065	0.6790	0.0848	12.2	-0.2133	0.6599	0.0894
11.6	-0.2038	0.6710	0.0874	12.4	-0.2106	0.6523	0.0918
11.8	-0.2013	0.6632	0.0899	12.6	-0.2081	0.6448	0.0942
12.0	-0.1988	0.6556	0.0923	12.8	-0.2057	0.6375	0.0965
12.2	-0.1965	0.6482	0.0947	13.0	-0.2034	0.6304	0.0987
12.4	-0.1944	0.6410	0.0969	13.2	-0.2013	0.6235	0.1009
12.6	-0.1923	0.6339	0.0991	13.4	-0.1992	0.6167	0.1030
12.8	-0.1903	0.6270	0.1012	13.6	-0.1973	0.6100	0.1051
13.0	-0.1884	0.6202	0.1033	13.8	-0.1954	0.6035	0.1070
13.2	-0.1866	0.6136	0.1053	14.0	-0.1936	0.5971	0.1090
13.4	-0.1849	0.6070	0.1072	14.2	-0.1919	0.5907	0.1109
13.6	-0.1832	0.6006	0.1092	14.4	-0.1902	0.5845	0.1127
13.8	-0.1816	0.5943	0.1110	14.6	-0.1886	0.5785	0.1145
14.0	-0.1801	0.5882	0.1128	14.8	-0.1871	0.5725	0.1163
14.2	-0.1786	0.5821	0.1146	15.0	-0.1856	0.5666	0.1180
14.4	-0.1772	0.5761	0.1164	15.2	-0.1842	0.5607	0.1197
14.6	-0.1758	0.5702	0.1181	15.4	-0.1828	0.5550	0.1214

CHAPTER 5

Reliability evaluation of project duration considering activity correlation

5.1 Introduction

In traditional methods for reliability evaluation of project duration, such as critical path method (CPM) [1-4], program evaluation and review technique (PERT) [5-6], the Monte Carlo simulation (MCS) [7-13], fast and accurate reliability bounds (FARB) [14]. etc., it is assumed that the durations of individual activities are independent. In real situations, however, factors such as weather, site conditions, and design changes can affect the duration of project activities. These factors usually affect multiple activities on a particular project and may cause the activity duration to be correlated [15-16]. Traditional methods will not capture the correlation that may exist between the durations of different activities in a project network [16-17].

A survey conducted by the Project Management Institute [18] in 1999 showed that nearly 20% of project management software supports MCS. Nowadays, with the rapid development of science and technology, it is easy to understand that the project management software supporting MCS should exceed 20%. For example, PerMaster [19] uses planning data in tools such as MS-Project and Primavera, combined with MCS to provide project risk analysis in terms of project completion time. However, MCS assumes the activity durations are independent. Therefore, PerMaster cannot deal with the reliability assessment of project duration caused by activity correlation. Then people who use this tool will ignore the relevance and even fail to understand correlations. Under these circumstances, the method of estimating the project duration lacks accuracy and is inconsistent with actual project practice. Therefore, dealing with the correlation between activities is very necessary for the reliability assessment of the project duration [20-21].

To solve this problem, an MCS based on fourth-moment transformation technique for reliability assessment method of project duration is proposed in this chapter. This chapter consists of five parts: firstly, after obtaining the first four moments and the correlation information of activity durations, the performance function $G(\mathbf{X})$ of the project duration is determined, where $\mathbf{X}$ is a vector of activity durations, some or all of which may be correlated. Secondly, the fourth-moment transformation approach for treating correlations in activity durations is developed. Thirdly, utilizing MCS to calculate the reliability of project duration performance function $G(\mathbf{X})$. Finally, the accuracy of the proposed method is demonstrated through a numerical example.

5.2 Correlation information acquisition and performance function establishment

5.2.1 Correlation information acquisition of the activity durations

Usually, when dealing with correlations among primary variables, it is only the linear correlations. If all the primary variables are normally distributed, then the linear correlations between variables are the true correlations. However, in general, all the primary variables are not subjected to normal distribution. Therefore, one is faced with the problem of non-linear correlations. It is still complex and largely unresolved theoretical issues of obtaining and treating non-linear correlations. Therefore, the correlations among primary activity durations are assumed linear. This assumption implies that relationships between activity durations are approximately linear.

In the planning stage of most engineering projects, there are large data limitations. However, the treatment of the correlation between variables is very important for the reliability assessment of the project duration. Many researchers have recognized this necessity [23-26]. Therefore, the correlation between the main variables must be derived subjectively by experts [22].

5.2.2 Performance function establishment of the project duration

As mentioned in chapter 3, for the project network, each failure path f_i can be determined by a performance function $g_i = g_i(\mathbf{X}) = t - T_i(\mathbf{X})$ [27], such that $f_i = (g_i \leq 0)$, and the failure probability of the project duration can be given as follows

$$\begin{aligned} P_F &= Prob[f_1 \cup f_2 \cup \ldots \cup f_L] \\ &= Prob[(g_1(\mathbf{X}) \leq 0) \cup (g_2(\mathbf{X}) \leq 0), \ldots, \cup (g_L(\mathbf{X}) \leq 0)] \end{aligned} \tag{5-1}$$

where t is the target project duration, $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$ indicates the vector of random variables, and T_i (**X**) denotes the duration of path i.

On the contrary, the reliability of the project duration is the probability that none of the L possible failure paths will occur, that is, the intersection of all the complements of the L potential failure paths, which gives

$$\begin{aligned} P_S &= Prob[\overline{f_1 \cup f_2 \cup ... \cup f_L}] = Prob[\overline{f_1} \cap \overline{f_2} \cap ... \cap \overline{f_L}]] \\ &= Prob[(g_1(\mathbf{X})>0) \cap (g_2(\mathbf{X})>0) \cap ... \cap (g_L(\mathbf{X})>0)] \end{aligned} \tag{5-2}$$

Eq. (5-2) means that the reliability of a project duration is the event that all the L performance functions have to be larger than zero, that is the target duration t is larger than all the path durations T_i(**X**), and it is equal to the event that the target duration t is larger than the maximum of T_i(**X**). This means

$$P_S = Prob\{t - \max[T_1(\mathbf{X}), T_2(\mathbf{X}),...,T_L(\mathbf{X})]] > 0\} \tag{5-3}$$

The corresponding failure probability of the project duration can be expressed as

$$P_F = Prob\{t - \max[T_1(\mathbf{X}), T_2(\mathbf{X}),...,T_L(\mathbf{X})] \le 0\} \tag{5-4}$$

Thus, the overall state performance function of the project duration, G, can be expressed as

$$G(\mathbf{X}) = t - \max[T_1(\mathbf{X}), T_2(\mathbf{X}),...,T_L(\mathbf{X})] \tag{5-5}$$

5.3 Fourth-moment normal transformation for correlated activity durations

In general, the random variable X_i can be standardized, such as:

$$X_{is} = (X_i - \mu_{X_i}) / \sigma_{X_i} \tag{5-6}$$

where μ_{Xi} and σ_{Xi} are the mean and standard deviation of X_i, respectively.

Consider correlated random variables X_i (i=1,..., n) with correlation matrix, if the first four moments (i.e., mean, standard deviation, skewness and kurtosis) of X_i are known, the standardized variable X_{is} of X_i can be expressed as:

$$X_{is} = S^{-1}(Z_i) = a_i + b_i Z_i + c_i Z_i^2 + d_i Z_i^3 \tag{5-7}$$

where Zi is the ith correlated standard normal random variable; $Sz(Zi)$ is the third-order polynomial of Zi; and ai, bi, ci, and di are the polynomial coefficients, which can be determined by making the first four central moments of $Sz(Zi)$ equal those of Xis [28], as shown in Appendix A.

According to Eqs. (5-6) and (5-7), the relation between *Xi* and *Zi* can be expressed as:

$$X_i = \mu_{X_i} + \sigma X_i (a_i + b_i Z_i + c_i Z_i^2 + d_i Z_i^3), (i = 1,\ 2,\ \ldots, n) \tag{5-8}$$

Assume that ρ_{ij} is the correlation coefficient between X_i and X_j , and ρ_{0ij} is the correlation coefficient between Z_i and Z_j, which referred as equivalent correlation coefficient, then according to the definition of correlation coefficient, the following can be gotten,

$$\rho_{ij} = E\left(X_{is} \cdot X_{js}\right) = \left(a_i, b_i, c_i, d_i\right) \mathbf{R} \left(a_j, b_j, c_j, d_j\right)^T \tag{5-9a}$$

where

$$\mathbf{R} = E\left[\left(1, Z_i, Z_i^2, Z_i^3\right)^T \cdot \left(1, Z_j, Z_j^2, Z_j^3\right)\right] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \rho_{0ij} & 0 & 3\rho_{0ij} \\ 1 & 0 & 2\rho_{0ij}^2 + 1 & 0 \\ 0 & 3\rho_{0ij} & 0 & 6\rho_{0ij}^3 + 9\rho_{0ij} \end{bmatrix} \tag{5-9b}$$

Substituting Eq. (5-9b) into Eq. (5-9a) the following can be gotten:

$$\rho_{ij} = (b_i b_j + 3d_i b_j + 3b_i d_j + 9d_i d_j)\rho_{0ij} + 2c_i c_j \rho_{0ij}^2 + 6d_i d_j \rho_{0ij}^3 \tag{5-10a}$$

It can be observed that ρ_{0ij} can be determined from solving Eq. (5-10a). It is worth noting that the valid solution of ρ_{0ij} should be restricted by the following conditions to satisfy the definition of the correlation coefficient:

$$-1 \le \rho_{0ij} \le 1,\ \rho_{ij} \cdot \rho_{0ij} \ge 0, \text{ and } \left|\rho_{0ij}\right| \ge \left|\rho_{ij}\right| \tag{5-10b}$$

Based on Eqs. (5-10a) and (5-10b), the expressions of the equivalent correlation coefficient ρ_{0ij} and the upper and lower bounds of original correlation coefficient ρ_{ij}, which ensure the transformation executable are summarized in Table 5-1 [29], more detail about the derivation process of Table 5-1 can be seen in Appendix B [29].

Table 5-1 Equivalent correlation coefficient and bounds for third-moment transformation

Conditions			Application range of ρ_{ij}	ρ_{0ij}
$c_i c_j = 0$			$[-b_i b_j,\ b_i b_j]$	$\dfrac{\rho_{ij}}{b_i b_j}$
$c_i c_j > 0$	$8c_i c_j - b_i^2 b_j^2 > 0$	$4c_i c_j - b_i b_j > 0$	$[-b_i^2 b_j^2 / (8c_i c_j), b_i b_j + 2c_i c_j]$	$\dfrac{-b_i b_j + \sqrt{b_i^2 b_j^2 + 8c_i c_j \rho_{ij}}}{4c_i c_j}$
$c_i c_j < 0$	$8c_i c_j + b_i^2 b_j^2 < 0$	$4c_i c_j + b_i b_j < 0$	$[2c_i c_j - b_i b_j, -b_i^2 b_j^2 / (8c_i c_j)]$	$\dfrac{-b_i b_j + \sqrt{b_i^2 b_j^2 + 8c_i c_j \rho_{ij}}}{4c_i c_j}$
Otherwise			$[2c_i c_j - b_i b_j, b_i b_j + 2c_i c_j]$	$\dfrac{-b_i b_j + \sqrt{b_i^2 b_j^2 + 8c_i c_j \rho_{ij}}}{4c_i c_j}$

Consider correlated random variables X_i ($i = 1, \ldots, n$) with correlation matrix C_X

$$\mathbf{C_X} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{12} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix} \tag{5-11}$$

where ρ_{ij} is the correlation coefficient between X_i and X_j.

If the first rth moments of **X** are known, the pair of rth moment of fourth-moment normal transformation for X_i can be given as:

$$X_{is} = S^{-1}(Z_i, \mathbf{M}) \tag{5-12a}$$

$$Z_i = S(X_{is}, \mathbf{M}) \tag{5-12b}$$

where Z_i is the rth moment of the fourth-moment standard normal variable.

Substituting Eq.(5-12a) into the definition of ρ_{ij}, one obtains

$$\rho_{ij} = E\left(X_{is} \cdot X_{js}\right) = E[S^{-1}(Z_i, \mathbf{M}) \cdot S^{-1}(Z_j, \mathbf{M})] \tag{5-13}$$

Since **X** are correlated random variables, **Z** are obviously also correlated random variables. Assuming the correlation coefficient between Z_i and Z_j is ρ_{0ij}, the relationship between ρ_{0ij} and ρ_{ij} can be defined by Eq. (5-13). Then, ρ_{0ij} can be determined from ρ_{ij} and **M**, and the correlation matrix of rth moment fourth-moment standard normal variables, $\mathbf{C_Z}$, can be written as:

$$\mathbf{C_Z} = \begin{bmatrix} 1 & \rho_{012} & \cdots & \rho_{01n} \\ \rho_{012} & 1 & \cdots & \rho_{02n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{0n1} & \rho_{0n2} & \cdots & 1 \end{bmatrix} \tag{5-14}$$

Using Cholesky decomposition, the correlation matrix $\mathbf{C_Z}$ can be rewritten as,

$$\mathbf{C_Z} = \mathbf{L}_0\mathbf{L}_0^T \tag{5-15}$$

where $\mathbf{L}_0$ is the lower triangular matrix from Cholesky decomposition and $\mathbf{L}_0^T$ is the transpose matrix of $\mathbf{L}_0$.

With the correlated rth moment fourth-moment standard normal vector **Z** obtained from Eq. (5-12b), the independent standard normal vector $\mathbf{U} = (U_1, U_2, \ldots, U_n)$ can then be given as:

$$\mathbf{U} = \mathbf{L}_0^{-1}\mathbf{Z} \tag{5-16}$$

In order to obtain the inverse transformation, the independent standard normal vector **U** is firstly transformed into correlated rth moment fourth-moment standard normal vector **Z** by

$$\mathbf{Z} = \mathbf{L}_0 \mathbf{U} \tag{5-17}$$

where $\mathbf{L}_0^{-1}$ is the inverse matrix of $\mathbf{L}_0$, and $\mathbf{L}_0$ is expressed as:

$$\mathbf{L}_0 = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \tag{5-18}$$

Using Eq. (5-12a), the reduced random vector of $\mathbf{X}$ can be obtained, and then the u-x_s transformation can be accomplished.

In particular, when both Eqs. (7.30a) and (7.30b) hold true only for the first two moments, i.e., the mean value and the standard deviation, Eqs. (5-12a) and (5-12b) become

$$X_{is} = Z_i \tag{5-19a}$$

$$Z_i = X_{is} \tag{5-19b}$$

In this case, $\rho_{0ij} = \rho_{ij}$.

Then, utilizing the Cholesky decomposition, the correlated fourth-moment standard variables can be converted to independent standard (reduced) space:

$$\mathbf{U} = \mathbf{L}_0^{-1} \mathbf{X}_S \tag{5-20a}$$

$$\mathbf{X}_S = \mathbf{L}_0 \mathbf{U} \tag{5-20b}$$

where $\mathbf{L}_0$ is the same as the matrix shown in Eq. (5-18), and $\mathbf{L}_0^{-1}$ is the inverse matrix of $\mathbf{L}_0$.

From Eqs. (5-17), (5-18), and (5-7), the u-x transformations can be expressed as

$$X_i = \mu_{X_i} + \sigma_{X_i}\left[a_i + b_i \sum_{k=1}^{i} l_{ik} U_k + c_i \left(\sum_{k=1}^{i} l_{ik} U_k\right)^2 + d_i \left(\sum_{k=1}^{i} l_{ik} U_k\right)^3\right], \quad (i = 1,\ 2,\ \cdots,\ n) \tag{5-21}$$

where l_{ik} is the ith row kth column element of matrix $\mathbf{L}_0$.

Based on Eq. (5-16) and (5-12b), the normal transformation based on the fourth moment transformation is expressed as

$$U_i = \sum_{k=1}^{i} h_{ik} S(X_{ks}) = \sum_{k=1}^{i} h_{ik} S\left[(X_k - \mu_{X_k}) / \sigma_{X_k}\right], \quad (i = 1,\ 2,\ \cdots,\ n) \tag{5-22}$$

where h_{ik} is the ith row kth column element of matrix $\mathbf{L}_0^{-1}$; and $S(X_{is})$ is given by Table 5-2.

Table 5-2 Complete expression of the fourth moment normal transformation of $S(X_{is})$

Parameters			Range of X_i	Expression of $S(X_{is})$	Type
$d_i < 0$			$J_2^* < G < J_1^*$	$-2r\cos[(\theta+\pi)/3]-a/3$	I
$d_i > 0$	$P < 0$	$\alpha_{3Xi} \geq 0$	$J_1^* < G < J_2^*$	$2r\cos(\theta/3)-a/3$	II
			$G \geq J_2^*$	$\sqrt[3]{A}+\sqrt[3]{B}-a/3$	
		$\alpha_{3Xi} < 0$	$J_1^* < G < J_2^*$	$-2r\cos[(\theta-\pi)/3]-a/3$	III
			$G \leq J_1^*$	$\sqrt[3]{A}+\sqrt[3]{B}-a/3$	
	$P \geq 0$		$(-\infty, \infty)$	$\sqrt[3]{A}+\sqrt[3]{B}-a/3$	VI
$a_4 = 0$		$\alpha_{3xi} \neq 0$	$b_i^2+4c_i(c_i+X_{is}) \geq 0$	$[-b_i+\sqrt{b_i^2+4c_i(c_i+X_{is})}]/2c_i$	V
		$\alpha_{3xi} = 0$	$(-\infty, \infty)$	X_{is}	IV

5.4 Evaluation of the project duration

Using the proposed fourth-moment transformation described above, the correlated nonnormal activity durations can be transformed into independent standard normal ones. Then Monte Carlo simulation (MCS) can be used to calculate the reliability of the project duration.

For a project duration performance function $G(\mathbf{X})$, randomly give a sample vector $\mathbf{x}_k$ for the activity duration vector $\mathbf{X}$, $G(\mathbf{x}_k) \leq 0$ is then checked. If the performance function is violated, the structure or structural element will be considered to fail. If the experiment is repeated for N times, the probability of failure of the project duration can be approximated by

$$P_F \approx \frac{n[G(\mathbf{x}_k) \leq 0]}{N} \tag{5-23}$$

where $n[G(\mathbf{x}_k) \leq 0]$ is the number of trials for which $G(\mathbf{x}_k) \leq 0$. Obviously, the number N of trials required is related to the desired accuracy for P_F.

Then the reliability can be calculated as follows:

$$P_s = 1 - P_F \tag{5-24}$$

5.5 Application of the proposed method

The proposed method is applied to a practical project from [31]. Activity network and descriptions are shown in Fig. 5-1. The first four moments of the activity duration are listed in Table 5-3. Activity information of network paths for the project is shown in

Table 5-4. The correlation of the activity durations is shown in Table 5-5. The target duration of this project is 90 days.

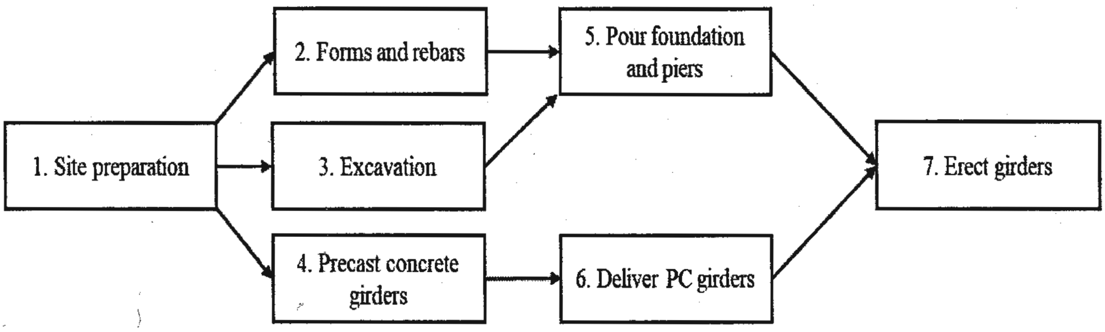


Fig. 5-1 Example network

Table 5-3 First four moments of activity duration

Activity i	Distribution	Mean	Standard deviation	Skewness	Kurtosis
1	Lognormal	14.1667	2.1667	0.2660	3.12605
2	Lognormal	15.3333	1.3333	0.3530	3.2223
3	Lognormal	15.5	2.1667	0.4221	3.31841
4	Lognormal	12.1667	1.3333	0.1970	3.06908
5	Lognormal	22.5	2.1667	0.2134	3.08113
6	Lognormal	14.1667	1.6667	0.2058	3.07539
7	Lognormal	9.1667	0.8333	0.2735	3.13326

Table 5-4 Activity network paths for the project

Path	Activities of each path
1	1, 2, 5, 7
2	1, 3, 5, 7
3	1, 4, 6, 7

Only consider the critical path, using MCS with 500 samples, the probability of failure P_f is obtained as 0.08. Compared with the results generated by Yang [32] and CPM, it is nearly the same. So that the accuracy of the proposed method is demonstrated. While as mentioned in chapter 3, the influence of paths correlation cannot be ignored. Thus, according to Eq. (5-5), the project duration performance function $G(\mathbf{X})$ can be defined:

$$G(\mathbf{X}) = 90 - \max[T_1(\mathbf{X}), T_2(\mathbf{X}), T_3(\mathbf{X})] \tag{5-25}$$

where

$$T_1(\mathbf{X}) = x_1 + x_2 + x_5 + x_7, \quad T_2(\mathbf{X}) = x_1 + x_3 + x_5 + x, \quad T_3(\mathbf{X}) = x_1 + x_4 + x_6 + x$$

Table 5-5 Correlation coefficient between activity duration

Activity i	ρ_{i1}	ρ_{i2}	ρ_{i3}	ρ_{i4}	ρ_{i5}	ρ_{i6}	ρ_{i7}
1	1	0	0	0	0	0	0
2	0	1	0	0	0.5	0	0
3	0	0	1	0	0	0	0
4	0	0	0	1	0	0.8	0.8
5	0	0	0	0	1	0	0
6	0	0	0	0	0	1	0.8
7	0	0	0	0	0	0	1

The correlation matrix of basic random variables is given as:

$$C_{\mathbf{X}} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0.5 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0.8 & 0 & 1 & \\ 0 & 0 & 0 & 0.8 & 0 & 0.8 & 1 \end{pmatrix}$$

Based on the fourth-moment transformation technique, the correlative matrix of standard normal variables can be gotten:

$$\rho_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.5013 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0.8004 & 0.8006 \\ 0 & 0.5013 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0.8004 & 0 & 1 & 0.8006 \\ 0 & 0 & 0 & 0.8006 & 0 & 0.8006 & 1 \end{pmatrix}$$

Then the lower triangular matrix L_0 can be obtained from Cholesky decomposition of C_Z:

$$L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.5013 & 0 & 0 & 0.8653 & 0 & 0 \\ 0 & 0 & 0 & 0.8004 & 0 & 0.5996 & 0 \\ 0 & 0 & 0 & 0.8006 & 0 & 0.26659 & 0.5366 \end{pmatrix}$$

Then the performance function $G(\mathbf{X})$ in Eq. (5-25) can be rewritten as $G(\mathbf{U})$, where $U_i (i = 1, 2, \ldots 7)$ are independent standard normal variables.

Then using the method of MCS with 100,000 samples, the probability of failure P_f is obtained as 0.05824. It can be seen that the result is very different from considering only the critical path.

5.6 Conclusions

In this chapter, a new project duration reliability method has been introduced to treat the correlations between activity durations. It has introduced a fourth-moment transformation technique for transforming the correlated nonnormal random variables into independent standard normal ones, in which the correlated nonnormal random variables are firstly transformed into correlated standard normal random ones using the fourth-moment transformation; and the correlated standard normal random variables are then transformed into independent standard normal random ones using Cholesky decomposition. Then Based on the proposed fourth-moment transformation, a MCS method for the reliability analysis of the project duration involving correlated random variables is conducted. A simple numeral example has demonstrated the accuracy of the proposed method.

Appendix A

According to Fleishman [28], the third-order polynomial transformation is as follows

$$X_{is} = S^{-1}(Z_i) = a_i + b_i Z_i + c_i Z_i^2 + d_i Z_i^3 \quad \text{(A-1)}$$

where X_s is the standardized random variable; U is the standard normal random variable, and a_1, a_2, a_3 and a_4 are the polynomial coefficients that can be determined by moment-matching method (Fleishman 1978), i.e., making the first four moments of the left side of Eq. (A-1) equal to those of the right side, i.e.,

$$a_i + c_i = 0 \tag{A-2a}$$

$$b_i^2 + 2c_i^2 + 6b_i d_i + 15d_i^2 = 1 \tag{A-2b}$$

$$6b_i^2 c_i + 8c_i^3 + 72b_i c_i d_i + 270c_i d_i^2 = \alpha_{3X_i} \tag{A-2c}$$

$$3(b_i^4 + 20b_i^3 d_i + 210b_i^2 d_i^2 + 1260b_i d_i^4 + 3465d_i^4) + 12c_i^2(5b_i^2 + 5c_i^2 + 78b_i d_i + 375d_i^2) = \alpha_{4X_i} \tag{A-2d}$$

Simplification of Eqs. (7.71a-d) leads the following equations of parameters a_2 and a_4

$$2A_1 A_2^2 = \alpha_{3X_i}^2 \tag{A-3a}$$

$$3A_1 A_3 + 3A_4 = \alpha_{4X_i} \tag{A-3b}$$

where

$$A_1 = 1 - b_i^2 - 6b_i d_i - 15d_i^2 \tag{A-3c}$$

$$A_2 = 2 + b_i^2 + 24b_i d_i + 105d_i^2 \tag{A-3d}$$

$$A_3 = 5 + 5b_i^2 + 126b_i d_i + 675d_i^2 \tag{A-3e}$$

$$A_4 = b_i^4 + 20b_i^3 d_i + 210b_i^2 d_i^2 + 1260b_i d_i^3 + 3465d_i^4 \tag{A-3f}$$

Since the values α_{3Xi} and α_{4Xi} are known, the parameters a_i and c_i can be readily solved as

$$c_i = -a_i = \frac{\alpha_{3X_i}}{2A_2} \tag{A-4}$$

When $|\alpha_{3Xi}| \le 2$ and $(4\alpha_{3Xi} + 7)/3 \le \alpha_{4Xi} \le 12$, the explicit expressions for the four coefficients suggested by Zhao and Lu [23] can be adopted, which are expressed as:

$$c_i = -a_i = \frac{\alpha_{3X_i}}{6(1+6k_i)},\ \mathrm{b}_i = \frac{1-3k_i}{1+a_i^2+k_i^2},\ d_i = \frac{k_i}{1+a_i^2+12k_i^2} \tag{A-5a}$$

$$k_i = \frac{1}{36}(\sqrt{6\alpha_{4X_i} - 8\alpha_{3Xi}^2 - 14} - 2) \tag{A-5b}$$

Appendix B

When $c_i c_j$ is not zero, Eq. (5-10a) is a quadratic equation. For brevity, the right side of Eq. (5-10a) is expressed as $h(\rho_{0ij})$, i.e.,

$$\rho_{ij} = h(\rho_{0ij}) = b_i b_j \cdot \rho_{0ij} + 2c_i c_j \cdot \rho_{0ij}^2 \tag{B-1}$$

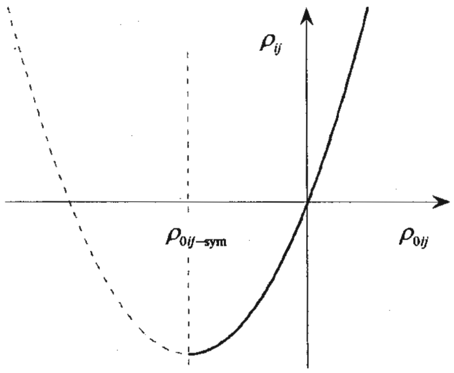
The axis of symmetry of $h(\rho_{0ij})$ is at:

$$\rho_{0ij} = \rho_{0ij-\text{sym}} = -b_i b_j / (4c_i c_j) \tag{B-2}$$

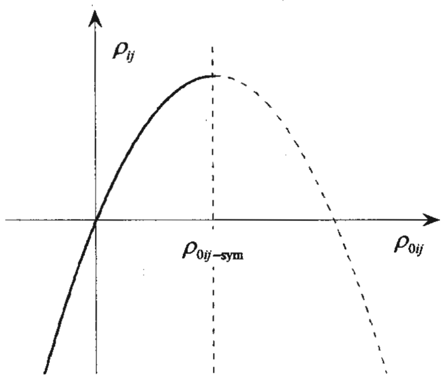
According to the third moment fourth-moment normal transformation, its region of application is given as:

$$-2\sqrt{2} \le \alpha_{3X_i} \le 2\sqrt{2} \text{ and } -2\sqrt{2} \le \alpha_{3X_j} \le 2\sqrt{2} \tag{B-3}$$

It can be observed that $b_i b_j$ is non-negative. Therefore, $\rho_{0ij\text{-sym}}$ is positive for $c_i c_j < 0$ and negative for $c_i c_j > 0$. The shape of $h(\rho_{0ij})$ for $c_i c_j \neq 0$ is presented in Fig. B-1, in which the solid lines denote the region satisfying the condition that $\rho_{0ij} \cdot \rho_{ij} \ge 0$ and ρ_{0ij} is an increasing function of ρ_{ij}.



(a) $c_i c_j > 0$



(b) $c_i c_j < 0$

Fig. B-1 The shape of $h(\rho_{0ij})$ for $c_i c_j \neq 0$

From Fig. B-1(a), the equivalent correlation coefficient ρ_{0ij} for $c_i c_j > 0$ can be given as:

$$\rho_{0ij} = \frac{-b_i b_j + \sqrt{b_i^2 b_j^2 + 8c_i c_j \rho_{ij}}}{4c_i c_j} \tag{B-4}$$

In order to satisfy $-1 \le \rho_{0ij} \le 1$, the ρ_{ij} should have application bounds, i.e., $\rho_{ij} \in [\rho_{ij-\min}, \rho_{ij-\max}]$, in which $\rho_{ij\text{-min}}$ and $\rho_{ij\text{-max}}$ are the lower and upper bounds, respectively. According to Fig. B-1(a) and $|\rho_{0ij}| \ge |\rho_{ij}|$, the upper bound, $\rho_{ij\text{-max}}$, can be readily given as:

$$\rho_{ij-\max} = h(1) = b_i b_j + 2c_i c_j \tag{B-5a}$$

And the lower bound can be determined as:

$$\rho_{ij-\min} = \begin{cases} h(\rho_{0ij-\text{sym}}), & h(\rho_{0ij-\text{sym}}) > -1 \text{ and } \rho_{0ij-\text{sym}} > -1 \\ h(-1), & \text{otherwise} \end{cases} \tag{B-5b}$$

where $h(-1) = 2c_i c_j - b_i b_j$; $h(\rho_{0ij\text{-sym}}) = -b_i^2 b_j^2 / (8c_i c_j)$. Thus, Eq. (B-5b) can be rewritten as:

$$\rho_{ij-\min} = \begin{cases} -b_i^2 b_j^2 / (8c_i c_j), & 8c_i c_j - b_i^2 b_j^2 > 0 \text{ and } 4c_i c_j - b_i b_j > 0 \\ 2c_i c_j - b_i b_j, & \text{otherwise} \end{cases} \tag{B-5c}$$

Similarly, ρ_{0ij} for $c_ic_j < 0$ is also given by Eq. (B-4), and the application bound of ρ_{ij} is expressed as:

$$\rho_{ij-\max} = \begin{cases} -b_i^2 b_j^2 / (8c_i c_j), & 8c_i c_j + b_i^2 b_j^2 < 0 \text{ and } 4c_i c_j + b_i b_j < 0 \\ b_i b_j + 2c_i c_j, & \text{otherwise} \end{cases} \quad \text{(B-6a)}$$

$$\rho_{ij-\min} = 2c_i c_j - b_i b_j \quad \text{(B-6b)}$$

When $c_ic_j = 0$, according to Eq. (5-10a), ρ_{0ij} can be readily determined as:

$$\rho_{0ij} = \rho_{ij} / b_i b_j \quad \text{(B-7)}$$

and the bound for $c_ic_j = 0$ can be determined as:

$$\rho_{ij-\max} = b_i b_j \quad \text{(B-8a)}$$

$$\rho_{ij-\max} = -b_i b_j \quad \text{(B-8b)}$$

Finally, the expressions are summarized in Table 5-1 for the equivalent correlation coefficient ρ_{0ij} and the upper and lower bounds of original correlation coefficient ρ_{ij} to ensure the transformation executable.

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CHAPTER 6

Conclusions

In this study, simple and effective methods based on the method of moments have been proposed to assess the reliability of the project duration. The contents of this paper are summarized as follows:

In chapter two, a new computing model combined with PERT and cubic normal distribution has been proposed. A real project was analyzed by comparing cubic normal distribution and normal distribution with Monte Carlo simulation which illustrated that the use of this new proposed model can enable more reasonable and accurate reliability analysis of the total construction duration for the projects.

In chapter three, a simple and effective method based on the method of moments has been proposed to assess the reliability of the project duration. The proposed method consists of three main sections: first, an overall performance function is established with respect to the project duration; second, the bivariate-dimension reduction is used to evaluate the first three moments of this performance function; third, the reliability of the project duration is assessed by the third-moment reliability index. Two numerical examples were used to demonstrate the efficiency and effectiveness of the proposed method. It can be found that: the proposed method can provide nearly the same reliability assessment results of the project duration compared with the MCS method with less calculation. And compared with the FARB method, the proposed method does not require the calculations of the correlation coefficients between any pair of paths and the joint failure probability of any pair of representative paths. In addition, the proposed method can give an explicit formula of project duration reliability curve under different target durations, avoiding repeated calculations as the target duration changes.

In chapter four, the reliability analysis method of construction schedule under the influence of single risk factor is proposed. The proposed method is based on cubic normal transformation, which utilizing the first four moments of the influencing factor in practical engineering was proposed. It can provide more accurate analysis result than other existing methods to analysis the reliability of construction schedule plan. Through case analysis, the proposed analysis method using cubic normal distribution is proved to be more reliable than that using the normal distribution.

In chapter five, an MCS based on fourth-moment transformation technique for reliability assessment method of project duration is proposed. It allows correlation between activity durations to be considered in network analysis. It has introduced a fourth-moment transformation technique for transforming the correlated nonnormal random variables into independent standard normal ones, in which the correlated nonnormal random variables are firstly transformed into correlated standard normal random ones using the fourth-moment transformation; and the correlated standard normal random variables are then transformed into independent standard normal random ones using Cholesky decomposition. Then Based on the proposed fourth-moment transformation, the MCS method for the reliability analysis of the project duration is conducted.

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